

HYDRODYNAMIC LIMIT WITH GEOMETRIC CORRECTION OF STATIONARY BOLTZMANN EQUATION

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ABSTRACT. We consider the hydrodynamic limit of a stationary Boltzmann equation in a unit plate with in-flow boundary. We prove the solution can be approximated in L^∞ by the sum of interior solution which satisfies steady incompressible Navier-Stokes-Fourier system, and boundary layer with geometric correction. Also, we construct a counterexample to the classical theory which states the behavior of solution near boundary can be described by the Knudsen layer derived from the Milne problem.

Keywords: ϵ -Milne problem, Boundary layer, Geometric correction.

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1. INTRODUCTION

1.1. Problem Formulation. We consider stationary Boltzmann equation for $F^\epsilon(\vec{x}, \vec{v})$ in a two-dimensional unit plate $\Omega = \{\vec{x} = (x_1, x_2) : |\vec{x}| \leq 1\}$ with velocity $\Sigma = \{\vec{v} = (v_1, v_2) \in \mathbb{R}^2\}$ as

$$\begin{cases} \epsilon \vec{v} \cdot \nabla_x F^\epsilon &= Q[F^\epsilon, F^\epsilon] \text{ in } \Omega \times \mathbb{R}^2, \\ F^\epsilon(\vec{x}_0, \vec{v}) &= B^\epsilon(\vec{x}_0, \vec{v}) \text{ for } \vec{x}_0 \in \partial\Omega \text{ and } \vec{n}(\vec{x}_0) \cdot \vec{v} < 0, \end{cases} \quad (1.1)$$

where $\vec{n}(\vec{x}_0)$ is the outward normal vector at \vec{x}_0 and the Knudsen number ϵ satisfies $0 < \epsilon \ll 1$. Here we have

$$Q[F, G] = \int_{\mathbb{R}^2} \int_{S^1} q(\vec{\omega}, |\vec{u} - \vec{v}|) \left(F(\vec{u}_*) G(\vec{v}_*) - F(\vec{u}) G(\vec{v}) \right) d\vec{\omega} d\vec{u}, \quad (1.2)$$

with

$$\vec{u}_* = \vec{u} + \vec{\omega} \left((\vec{v} - \vec{u}) \cdot \vec{\omega} \right), \quad \vec{v}_* = \vec{v} - \vec{\omega} \left((\vec{v} - \vec{u}) \cdot \vec{\omega} \right), \quad (1.3)$$

and the hard-sphere collision kernel

$$q(\vec{\omega}, |\vec{u} - \vec{v}|) = q_0 |\vec{u} - \vec{v}| |\cos \phi|, \quad (1.4)$$

for positive constant q_0 related to the size of ball, $\vec{\omega} \cdot (\vec{v} - \vec{u}) = |\vec{v} - \vec{u}| \cos \phi$ and $0 \leq \phi \leq \pi/2$. We intend to study the behavior of F^ϵ as $\epsilon \rightarrow 0$.

Based on the flow direction, we can divide the boundary $\gamma = \{(\vec{x}_0, \vec{v}) : \vec{x}_0 \in \partial\Omega\}$ into the in-flow boundary γ_- , the out-flow boundary γ_+ , and the grazing set γ_0 as

$$\gamma_- = \{(\vec{x}_0, \vec{v}) : \vec{x}_0 \in \partial\Omega, \vec{v} \cdot \vec{n}(\vec{x}_0) < 0\}, \quad (1.5)$$

$$\gamma_+ = \{(\vec{x}_0, \vec{v}) : \vec{x}_0 \in \partial\Omega, \vec{v} \cdot \vec{n}(\vec{x}_0) > 0\}, \quad (1.6)$$

$$\gamma_0 = \{(\vec{x}_0, \vec{v}) : \vec{x}_0 \in \partial\Omega, \vec{v} \cdot \vec{n}(\vec{x}_0) = 0\}. \quad (1.7)$$

It is easy to see $\gamma = \gamma_+ \cup \gamma_- \cup \gamma_0$. Hence, the boundary condition is only given on γ_- .

We assume that the boundary data can be expanded as

$$B^\epsilon(\vec{x}_0, \vec{v}) = \mu + \sqrt{\mu} b^\epsilon(\vec{x}_0, \vec{v}) = \mu + \sqrt{\mu} \left(\sum_{k=1}^{\infty} \epsilon^k b_k(\vec{x}_0, \vec{v}) \right), \quad (1.8)$$

where the standard Maxwellian is defined as

$$\mu(\vec{v}) = \frac{1}{2\pi} \exp \left(-\frac{|\vec{v}|^2}{2} \right), \quad (1.9)$$

and $b_k(\vec{x}_0, \vec{v})$ is independent of ϵ . We further require

$$\left| \langle \vec{v} \rangle^\vartheta e^{\zeta |\vec{v}|^2} \frac{B^\epsilon - \mu}{\sqrt{\mu}} \right|_{L^\infty} \leq C_0 \epsilon, \quad (1.10)$$

for $0 \leq \zeta \leq 1/4$ and $\vartheta \geq 3$, where $C_0 > 0$ is sufficiently small. The solution F^ϵ can be expressed as a perturbation of the standard Maxwellian

$$F^\epsilon(\vec{x}, \vec{v}) = \mu + \sqrt{\mu} f^\epsilon(\vec{x}, \vec{v}). \quad (1.11)$$

Then f^ϵ satisfies the equation

$$\begin{cases} \epsilon \vec{v} \cdot \nabla_x f^\epsilon + \mathcal{L}[f^\epsilon] &= \Gamma[f^\epsilon, f^\epsilon], \\ f^\epsilon(\vec{x}_0, \vec{v}) &= b^\epsilon(\vec{x}_0, \vec{v}) \text{ for } \vec{n} \cdot \vec{v} < 0 \text{ and } \vec{x}_0 \in \partial\Omega, \end{cases} \quad (1.12)$$

where

$$\Gamma[f^\epsilon, f^\epsilon] = \frac{1}{\sqrt{\mu}} Q[\sqrt{\mu} f^\epsilon, \sqrt{\mu} f^\epsilon], \quad (1.13)$$

$$\mathcal{L}[f^\epsilon] = -\frac{2}{\sqrt{\mu}} Q[\mu, \sqrt{\mu} f^\epsilon] = \nu(\vec{v}) f^\epsilon - K[f^\epsilon], \quad (1.14)$$

$$\nu(\vec{v}) = \int_{\mathbb{R}^2} \int_{S^1} q(\vec{v} - \vec{u}, \vec{\omega}) \mu(\vec{u}) d\vec{\omega} d\vec{u} \quad (1.15)$$

$$\begin{aligned} K[f^\epsilon](\vec{v}) &= K_2[f^\epsilon](\vec{v}) - K_1[f^\epsilon](\vec{v}) \\ &= \int_{\mathbb{R}^2} k(\vec{u}, \vec{v}) f^\epsilon(\vec{u}) d\vec{u} = \int_{\mathbb{R}^2} k_2(\vec{u}, \vec{v}) f^\epsilon(\vec{u}) d\vec{u} - \int_{\mathbb{R}^2} k_1(\vec{u}, \vec{v}) f^\epsilon(\vec{u}) d\vec{u}, \end{aligned} \quad (1.16)$$

$$K_1[f^\epsilon](\vec{v}) = \sqrt{\mu(\vec{v})} \int_{\mathbb{R}^2} \int_{S^1} q(\vec{v} - \vec{u}, \vec{\omega}) \sqrt{\mu(\vec{u})} f^\epsilon(\vec{u}) d\vec{\omega} d\vec{u}, \quad (1.17)$$

$$K_2[f^\epsilon](\vec{v}) = \int_{\mathbb{R}^2} \int_{S^1} q(\vec{v} - \vec{u}, \vec{\omega}) \sqrt{\mu(\vec{u})} \left(\sqrt{\mu(\vec{v}_*)} f^\epsilon(\vec{u}_*) + \sqrt{\mu(\vec{u}_*)} f^\epsilon(\vec{v}_*) \right) d\vec{\omega} d\vec{u}. \quad (1.18)$$

1.2. History. Hydrodynamic limit is one of the main steps to tackle Hilbert's Six Problems, i.e. to derive macroscopic physical rules from elementary atomic laws, especially, Newton's law in the framework of classical mechanics. In particular, it is well known the fluid-type equations can be formally derived from Boltzmann equation by applying Hilbert expansion to obtain the leading order term. A lot of works of evolutionary Boltzmann equation in the whole space domain have been presented (see [13], [15], [2], [3], [4], [5]) for either smooth solutions or renormalized solutions.

Unfortunately, despite its importance in both theory and practice, much less results are known for steady problem in a bounded domain, as [11] pointed out. In [8] and [9], the hydrodynamic limit of stationary Boltzmann equation with diffusive boundary were studied, both of which do not involve boundary layer approximation. However, for general boundary conditions, the presence of boundary layer effects is inevitable. It is noticeable that a lot of works, even for evolutionary Boltzmann equation in a bounded domain, cite the results from the classical paper [6] and take boundary layer analysis as being completely solved (see [16], [17]). Surprisingly, in [18], it was shown the boundary layer approximation based on Milne problem as in [6] breaks down both in the proof and result, due to intrinsic singularity of normal derivative. Therefore, any results on kinetic equations involving boundary layer effects from [6] should be reexamined.

In this paper, we will give a complete asymptotic analysis of stationary Boltzmann equations with in-flow boundary and construct a boundary layer with geometric correction, which has been proved to be effective in neutron transport equation by [18].

1.3. Main Theorem.

Theorem 1.1. *For given $B^\epsilon > 0$ satisfying (1.10) and $0 < \epsilon \ll 1$, there exists a unique positive solution $F^\epsilon = \mu + \sqrt{\mu} f^\epsilon$ to the stationary Boltzmann equation (1.1), where*

$$f^\epsilon = \epsilon^3 R_N + \left(\sum_{k=1}^N \epsilon^k \mathcal{F}_k^\epsilon \right) + \left(\sum_{k=1}^N \epsilon^k \mathcal{F}_k^\epsilon \right), \quad (1.19)$$

for $N \geq 3$, R_N satisfies (5.2), \mathcal{F}_k^ϵ and \mathcal{F}_k^ϵ satisfy (3.60) and (3.53). Also, there exists a $C > 0$ such that f^ϵ satisfies

$$\left\| \langle \vec{v} \rangle^\vartheta e^{\zeta |\vec{v}|^2} f^\epsilon \right\|_{L^\infty} \leq C\epsilon, \quad (1.20)$$

for any $\vartheta > 2$, $0 \leq \zeta \leq 1/4$.

Theorem 1.2. *For given $B^\epsilon > 0$ satisfying (1.10) with*

$$\frac{b_1}{\sqrt{\mu}} = \left(v_\phi e^{-(v_\phi^2 - 1) - M v_\eta^2} \right) = h(v_\eta, v_\phi), \quad (1.21)$$

where v_η and v_ϕ are defined as in (3.34) and M is sufficiently large such that

$$h(0, 1) = 1, \quad (1.22)$$

$$|h|_{L^2_-} \ll 1, \quad (1.23)$$

there exists $C > 0$ such that

$$\|f^\epsilon - (\mathcal{F}_1 + \mathcal{F}_1)\|_{L^\infty} \geq C\epsilon, \quad (1.24)$$

where the interior solution \mathcal{F}_1 is defined in (6.44) and boundary layer \mathcal{F}_1 is defined in (6.38).

1.4. Notation and Structure of This Paper. Throughout this paper, $C > 0$ denotes a constant that only depends on the parameter Ω , but does not depend on the data. It is referred as universal and can change from one inequality to another. When we write $C(z)$, it means a certain positive constant depending on the quantity z . We write $a \lesssim b$ to denote $a \leq Cb$.

Our paper is organized as follows: in Section 2, we establish the L^∞ well-posedness of the linearized Boltzmann equation; in Section 3, we present the asymptotic analysis of the equation (1.12); in Section 4, we prove the well-posedness and decay of the ϵ -Milne problem with geometric correction; in Section 5, we prove Theorem 1.1; finally, in Section 6, we show the classical approach and prove Theorem 1.2.

2. LINEARIZED STATIONARY BOLTZMANN EQUATION

We consider the linearized stationary Boltzmann equation

$$\begin{cases} \epsilon \vec{v} \cdot \nabla_x f + \mathcal{L}[f] &= S(\vec{x}, \vec{v}) \text{ in } \Omega, \\ f(\vec{x}_0, \vec{v}) &= h(\vec{x}_0, \vec{v}) \text{ for } \vec{x}_0 \in \partial\Omega \text{ and } \vec{v} \cdot \vec{n} < 0. \end{cases} \quad (2.1)$$

Let $\langle \cdot, \cdot \rangle$ be the standard L^2 inner product in $\Omega \times \mathbb{R}^2$. We define the L^p and L^∞ norms in $\Omega \times \mathbb{R}^2$ as usual:

$$\|f\|_{L^p} = \left(\int_{\Omega} \int_{\mathbb{R}^2} |f(\vec{x}, \vec{v})|^p d\vec{v} d\vec{x} \right)^{1/p}, \quad (2.2)$$

$$\|f\|_{L^\infty} = \sup_{(\vec{x}, \vec{v}) \in \Omega \times \mathbb{R}^2} |f(\vec{x}, \vec{v})|. \quad (2.3)$$

Define $d\gamma = |\vec{v} \cdot \vec{n}| d\varpi d\vec{v}$ on the boundary $\partial\Omega \times \mathbb{R}^2$ for ϖ as the surface measure. Define the L^p and L^∞ norms on the boundary as follows:

$$\|f\|_{L^p} = \left(\iint_{\gamma} |f(\vec{x}, \vec{v})|^p d\gamma \right)^{1/p}, \quad (2.4)$$

$$\|f\|_{L^p_{\pm}} = \left(\iint_{\gamma_{\pm}} |f(\vec{x}, \vec{v})|^p d\gamma \right)^{1/p}, \quad (2.5)$$

$$\|f\|_{L^\infty} = \sup_{(\vec{x}, \vec{v}) \in \gamma} |f(\vec{x}, \vec{v})|, \quad (2.6)$$

$$\|f\|_{L^\infty_{\pm}} = \sup_{(\vec{x}, \vec{v}) \in \gamma_{\pm}} |f(\vec{x}, \vec{v})|. \quad (2.7)$$

Also, we define

$$\|f\|_{L^2_v} = \|\sqrt{v}f\|_{L^2}. \quad (2.8)$$

Denote the Japanese bracket as

$$\langle \vec{v} \rangle = \sqrt{1 + |\vec{v}|^2} \quad (2.9)$$

Define the kernel operator \mathbb{P} as

$$\mathbb{P}[f] = \sqrt{\mu} \left(a_f(t, \vec{x}) + \vec{v} \cdot \vec{b}_f(t, \vec{x}) + \frac{|\vec{v}|^2 - 2}{2} c_f(t, \vec{x}) \right), \quad (2.10)$$

and the non-kernel operator $\mathbb{I} - \mathbb{P}$ as

$$(\mathbb{I} - \mathbb{P})[f] = f - \mathbb{P}[f]. \quad (2.11)$$

with

$$\int_{\mathbb{R}^2} (\mathbb{I} - \mathbb{P})[f] \begin{pmatrix} 1 \\ \vec{v} \\ |\vec{v}|^2 \end{pmatrix} d\vec{v} = 0 \quad (2.12)$$

Our analysis is based on the ideas in [8, 14].

2.1. Preliminaries.

Lemma 2.1. (*Green's Identity*) Assume $f(\vec{x}, \vec{v}), g(\vec{x}, \vec{v}) \in L^2(\Omega \times \mathbb{R}^2)$ and $\vec{v} \cdot \nabla_x f, \vec{v} \cdot \nabla_x g \in L^2(\Omega \times \mathbb{R}^2)$ with $f, g \in L^2(\gamma)$. Then

$$\iint_{\Omega \times \mathbb{R}^2} \left((\vec{v} \cdot \nabla_x f)g + (\vec{v} \cdot \nabla_x g)f \right) d\vec{x} d\vec{v} = \int_{\gamma_+} f g d\gamma - \int_{\gamma_-} f g d\gamma. \quad (2.13)$$

Proof. See the proof of [8, Lemma 2.2]. \square

Lemma 2.2. For any $\lambda > 0$, there exists a unique solution $f_\lambda(\vec{x}, \vec{v}) \in L^\infty(\Omega \times \mathbb{R}^2)$ to the penalized transport equation

$$\begin{cases} \lambda f_\lambda + \epsilon \vec{v} \cdot \nabla_x f_\lambda &= S(\vec{x}, \vec{v}) \text{ in } \Omega, \\ f_\lambda(\vec{x}_0, \vec{v}) &= h(\vec{x}_0, \vec{v}) \text{ for } \vec{x}_0 \in \partial\Omega \text{ and } \vec{v} \cdot \vec{n} < 0, \end{cases} \quad (2.14)$$

such that

$$\left\| \langle \vec{v} \rangle^\vartheta e^{\zeta |\vec{v}|^2} f_\lambda \right\|_{L^\infty} + \left| \langle \vec{v} \rangle^\vartheta e^{\zeta |\vec{v}|^2} f_\lambda \right|_{L^\infty_+} \leq C \left(\frac{1}{\lambda} \left\| \langle \vec{v} \rangle^\vartheta e^{\zeta |\vec{v}|^2} S \right\|_{L^\infty} + \left| \langle \vec{v} \rangle^\vartheta e^{\zeta |\vec{v}|^2} h \right|_{L^\infty_-} \right), \quad (2.15)$$

for all $\vartheta \geq 0, \zeta \geq 0$, and

$$\|f_\lambda\|_{L^2}^2 + \frac{\epsilon}{\lambda} \|f_\lambda\|_{L^2_+}^2 \leq C \left(\frac{1}{\lambda^2} \|S\|_{L^2}^2 + \frac{\epsilon}{\lambda} \|h\|_{L^2_-}^2 \right). \quad (2.16)$$

Proof. The characteristics $(X(s), V(s))$ of the equation (2.14) which goes through (\vec{x}, \vec{v}) is defined by

$$\begin{cases} (X(0), V(0)) &= (\vec{x}, \vec{v}) \\ \frac{dX(s)}{ds} &= \epsilon V(s), \\ \frac{dV(s)}{ds} &= 0. \end{cases} \quad (2.17)$$

which implies

$$\begin{cases} X(s) &= \vec{x} + \epsilon s \vec{v} \\ V(s) &= \vec{v} \end{cases} \quad (2.18)$$

Define the backward exit time $t_b(\vec{x}, \vec{v})$ and backward exit position $\vec{x}_b(\vec{x}, \vec{v})$ as

$$t_b(\vec{x}, \vec{v}) = \inf\{t > 0 : \vec{x} - \epsilon t \vec{v} \notin \Omega\}, \quad (2.19)$$

$$\vec{x}_b(\vec{x}, \vec{v}) = \vec{x} - \epsilon t_b(\vec{x}, \vec{v}) \vec{v} \notin \Omega. \quad (2.20)$$

Hence, we can rewrite the equation (2.14) along the characteristics as

$$\langle \vec{v} \rangle^\vartheta e^{\zeta |\vec{v}|^2} f_\lambda(\vec{x}, \vec{v}) = \langle \vec{v} \rangle^\vartheta e^{\zeta |\vec{v}|^2} h(\vec{x}_b, \vec{v}) e^{-\lambda t_b} + \int_0^{t_b} \langle \vec{v} \rangle^\vartheta e^{\zeta |\vec{v}|^2} S(\vec{x}_b + \epsilon s \vec{v}, \vec{v}) e^{-\lambda(t_b-s)} ds. \quad (2.21)$$

Then we can naturally estimate

$$\begin{aligned} \left\| \langle \vec{v} \rangle^\vartheta e^{\zeta |\vec{v}|^2} f_\lambda \right\|_{L^\infty} &\leq e^{-\lambda t_b} \left| \langle \vec{v} \rangle^\vartheta e^{\zeta |\vec{v}|^2} h \right|_{L^\infty_-} + \frac{1 - e^{-\lambda t_b}}{\lambda} \left\| \langle \vec{v} \rangle^\vartheta e^{\zeta |\vec{v}|^2} S \right\|_{L^\infty} \\ &\leq \left| \langle \vec{v} \rangle^\vartheta e^{\zeta |\vec{v}|^2} h \right|_{L^\infty_-} + \frac{1}{\lambda} \left\| \langle \vec{v} \rangle^\vartheta e^{\zeta |\vec{v}|^2} S \right\|_{L^\infty}, \end{aligned} \quad (2.22)$$

which further implies

$$\left| \langle \vec{v} \rangle^\vartheta e^{\zeta |\vec{v}|^2} f_\lambda \right|_{L^\infty_+} \leq \left| \langle \vec{v} \rangle^\vartheta e^{\zeta |\vec{v}|^2} h \right|_{L^\infty_-} + \frac{1}{\lambda} \left\| \langle \vec{v} \rangle^\vartheta e^{\zeta |\vec{v}|^2} S \right\|_{L^\infty}. \quad (2.23)$$

Since f_λ can be explicitly traced back to the boundary data, the existence naturally follows from above estimate. The uniqueness and L^2 estimates follow from Green's identity and $\|f_\lambda\|_{L^2} \leq C \left\| \langle \vec{v} \rangle^\vartheta e^{\zeta |\vec{v}|^2} f_\lambda \right\|_{L^\infty}$. \square

2.2. L^2 Estimates of Linearized Stationary Boltzmann Equation.

Lemma 2.3. *For any $\lambda > 0$, $m > 0$, there exists a unique solution $f_{\lambda,m}(\vec{x}, \vec{v}) \in L^2(\Omega \times \mathbb{R}^2)$ to the equation*

$$\begin{cases} \lambda f_{\lambda,m} + \epsilon \vec{v} \cdot \nabla_x f_{\lambda,m} + \mathcal{L}_m[f_{\lambda,m}] &= S(\vec{x}, \vec{v}) \text{ in } \Omega, \\ f_{\lambda,m}(\vec{x}_0, \vec{v}) &= h(\vec{x}_0, \vec{v}) \text{ for } \vec{x}_0 \in \partial\Omega \text{ and } \vec{v} \cdot \vec{n} < 0, \end{cases} \quad (2.24)$$

with \mathcal{L}_m the linearized Boltzmann operator corresponding to the cut-off cross section $q_m = \min\{q, m\}$. Also, the solution satisfies

$$\epsilon \|f_{\lambda,m}\|_{L^2_+}^2 + |f_{\lambda,m}|_{L^2_+}^2 \leq C(\lambda, m) \left(\|S\|_{L^2}^2 + \epsilon |h|_{L^2_-}^2 \right). \quad (2.25)$$

Proof. We divide the proof into several steps:

Step 1: Definition of iteration.

Denote $\mathcal{L}_m = \nu_m - K_m$. We define the iteration in l : $f_{\lambda,m}^0 = 0$ and for $l \geq 0$,

$$\begin{cases} \lambda f_{\lambda,m}^{l+1} + \epsilon \vec{v} \cdot \nabla_x f_{\lambda,m}^{l+1} + (1+M)\nu_m f_{\lambda,m}^{l+1} &= S(\vec{x}, \vec{v}) - (K_m + M\nu_m)[f_{\lambda,m}^l], \\ f_{\lambda,m}^{l+1}(\vec{x}_0, \vec{v}) &= h(\vec{x}_0, \vec{v}) \text{ for } \vec{x}_0 \in \partial\Omega \text{ and } \vec{v} \cdot \vec{n} < 0, \end{cases} \quad (2.26)$$

where $M > 0$ is a fixed real number to be determined later. Since

$$\|(K_m + M\nu_m)[f_{\lambda,m}^l]\|_{L^\infty} \leq C(m, M) \|f_{\lambda,m}^l\|_{L^\infty}, \quad (2.27)$$

Lemma 2.2 implies $f_{\lambda,m}^l \in L^\infty(\Omega \times \mathbb{R}^2)$ are well-defined for $l \geq 0$. However, we cannot directly obtain the existence of limit $f_{\lambda,m}^l$ as $l \rightarrow \infty$.

Step 2: The limit $l \rightarrow \infty$.

Based on Green's identity in Lemma 2.1, multiplying $f_{\lambda,m}^{l+1}$ on both sides of (2.26), we have

$$\begin{aligned} & \lambda \|f_{\lambda,m}^{l+1}\|_{L^2}^2 + \frac{\epsilon}{2} |f_{\lambda,m}^{l+1}|_{L^2_+}^2 + \langle (1+M)\nu_m f_{\lambda,m}^{l+1}, f_{\lambda,m}^{l+1} \rangle \\ &= \langle (K_m + M\nu_m)[f_{\lambda,m}^l], f_{\lambda,m}^{l+1} \rangle + \frac{\epsilon}{2} |h|_{L^2_-}^2 + \langle f_{\lambda,m}^{l+1}, S \rangle. \end{aligned} \quad (2.28)$$

Since $\mathcal{L}_m = \nu_m - K_m$ is a non-negative symmetric operator, we can always find M sufficiently large such that $K_m + M\nu_m$ is also a non-negative operator. Then we deduce

$$\begin{aligned} & \langle (K_m + M\nu_m)[f_{\lambda,m}^l], f_{\lambda,m}^{l+1} \rangle \\ & \leq \sqrt{\langle (K_m + M\nu_m)[f_{\lambda,m}^l], f_{\lambda,m}^l \rangle} \sqrt{\langle (K_m + M\nu_m)[f_{\lambda,m+1}^{l+1}], f_{\lambda,m}^{l+1} \rangle} \\ & \leq \frac{1}{2} \left(\langle (K_m + M\nu_m)[f_{\lambda,m}^l], f_{\lambda,m}^l \rangle + \langle (K_m + M\nu_m)[f_{\lambda,m+1}^{l+1}], f_{\lambda,m}^{l+1} \rangle \right) \\ & \leq \frac{1}{2} \left(\langle (1+M)\nu_m[f_{\lambda,m}^l], f_{\lambda,m}^l \rangle + \langle (1+M)\nu_m[f_{\lambda,m}^{l+1}], f_{\lambda,m}^{l+1} \rangle \right). \end{aligned} \quad (2.29)$$

Considering the fact

$$\langle (1+M)\nu_m[f_{\lambda,m}^l], f_{\lambda,m}^l \rangle \leq (1+M)m \|f_{\lambda,m}^l\|_{L^2}^2, \quad (2.30)$$

we obtain

$$\begin{aligned} & \left(\frac{\lambda}{(1+M)m} + 1 \right) \langle (1+M)\nu_m[f_{\lambda,m}^{l+1}], f_{\lambda,m}^{l+1} \rangle + \frac{\epsilon}{2} |f_{\lambda,m}^{l+1}|_{L^2_+}^2 \\ & \leq \frac{1}{2} \left(\langle (1+M)\nu_m[f_{\lambda,m}^l], f_{\lambda,m}^l \rangle + \langle (1+M)\nu_m[f_{\lambda,m}^{l+1}], f_{\lambda,m}^{l+1} \rangle \right) \\ & \quad + \frac{\epsilon}{2} |h|_{L^2_-}^2 + \frac{\lambda}{2(1+M)m} \langle (1+M)\nu_m[f_{\lambda,m}^{l+1}], f_{\lambda,m}^{l+1} \rangle + \frac{(1+M)m}{2\lambda} \|S\|_{L^2}^2. \end{aligned} \quad (2.31)$$

Since

$$\frac{\lambda}{(1+M)m} + 1 - \frac{1}{2} - \frac{\lambda}{2(1+M)m} > \frac{1}{2}, \quad (2.32)$$

by iteration over l , for

$$C_1(\lambda, m) = \frac{1}{1 + \frac{\lambda}{(1+M)m}} < 1, \quad (2.33)$$

$$C_2(\lambda, m) = \frac{1}{1 + \frac{\lambda}{(1+M)m}} \frac{(1+M)m}{\lambda} > 0, \quad (2.34)$$

we have

$$\begin{aligned} & \epsilon |f_{\lambda, m}^{l+1}|_{L_+^2}^2 + \left(1 + \frac{\lambda}{(1+M)m}\right) \langle (1+M)\nu_m[f_{\lambda, m}^{l+1}], f_{\lambda, m}^{l+1} \rangle \\ & \leq C_1(\lambda, m) \left(\epsilon |f_{\lambda, m}^l|_{L_+^2}^2 + \left(1 + \frac{\lambda}{(1+M)m}\right) \langle (1+M)\nu_m[f_{\lambda, m}^l], f_{\lambda, m}^l \rangle \right) \\ & \quad + C_2(\lambda, m) \left(\|S\|_{L^2}^2 + \epsilon |h|_{L_-^2}^2 \right). \end{aligned} \quad (2.35)$$

Taking the difference of $f_{\lambda, m}^{l+1} - f_{\lambda, m}^l$, we conclude that $f_{\lambda, m}^l$ is a Cauchy sequence. We take $l \rightarrow \infty$ to obtain $f_{\lambda, m}$ as a solution to the equation (2.24) satisfying

$$\epsilon |f_{\lambda, m}|_{L_+^2}^2 + \left(1 + \frac{\lambda}{(1+M)m}\right) \langle (1+M)\nu_m[f_{\lambda, m}], f_{\lambda, m} \rangle \leq \frac{C_2(\lambda, m)}{1 - C_1(\lambda, m)} \left(\|S\|_{L^2}^2 + \epsilon |h|_{L_-^2}^2 \right). \quad (2.36)$$

Then our results naturally follows. \square

Note that the estimate is not uniform in λ as $\lambda \rightarrow 0$. We need to find a stronger estimate of $f_{\lambda, m}$.

Lemma 2.4. *The solution $f_{\lambda, m}$ to the equation (2.24) satisfies the estimate*

$$\epsilon \|\mathbb{P}[f_{\lambda, m}]\|_{L^2} \leq C \left(\epsilon |f_{\lambda, m}|_{L_+^2} + \|(\mathbb{I} - \mathbb{P})[f_{\lambda, m}]\|_{L^2} + \|S\|_{L^2} + \epsilon |h|_{L_-^2} \right), \quad (2.37)$$

for $0 \leq \lambda \leq \epsilon \ll 1$.

Proof. Applying Green's identity in Lemma 2.1 to the equation (2.24), for any $\psi \in L^2(\Omega \times \mathbb{R}^2)$ satisfying $\vec{v} \cdot \nabla_x \psi \in L^2(\Omega \times \mathbb{R}^2)$ and $\psi \in L^2(\gamma)$, we have

$$\begin{aligned} & \lambda \iint_{\Omega \times \mathbb{R}^2} f_{\lambda, m} \psi + \epsilon \int_{\gamma_+} f_{\lambda, m} \psi d\gamma - \epsilon \int_{\gamma_-} f_{\lambda, m} \psi d\gamma - \epsilon \iint_{\Omega \times \mathbb{R}^2} (\vec{v} \cdot \nabla_x \psi) f_{\lambda, m} \\ & = - \iint_{\Omega \times \mathbb{R}^2} \psi (\mathbb{I} - \mathbb{P})[f_{\lambda, m}] + \iint_{\Omega \times \mathbb{R}^2} S \psi. \end{aligned} \quad (2.38)$$

Since

$$\mathbb{P}[f] = \sqrt{\mu} \left(a + \vec{v} \cdot \vec{b} + \frac{|\vec{v}|^2 - 2}{2} c \right), \quad (2.39)$$

our goal is to choose a particular test function ψ to estimate a , \vec{b} and c .

Step 1: Estimates of c .

We choose the test function

$$\psi = \psi_c = \sqrt{\mu(\vec{v})} \left(|\vec{v}|^2 - \beta_c \right) (\vec{v} \cdot \nabla_x \phi_c(\vec{x})), \quad (2.40)$$

where

$$\begin{cases} -\Delta_x \phi_c(\vec{x}) &= c(\vec{x}) \text{ in } \Omega, \\ \phi_c &= 0 \text{ on } \partial\Omega, \end{cases} \quad (2.41)$$

and β_c is a real number to be determined later. Based on the standard elliptic estimates, we have

$$\|\phi_c\|_{H^2} \leq C \|c\|_{L^2}. \quad (2.42)$$

With the choice of (2.40), the right-hand side(RHS) of (2.38) is bounded by

$$\text{RHS} \leq C \|c\|_{L^2} \left(\|(\mathbb{I} - \mathbb{P})[f_{\lambda,m}]\|_{L^2} + \|S\|_{L^2} \right). \quad (2.43)$$

We have

$$\vec{v} \cdot \nabla_x \psi_c = \sqrt{\mu(\vec{v})} \sum_{i,j=1}^2 \left(|\vec{v}|^2 - \beta_c \right) v_i v_j \partial_{ij} \phi_c(\vec{x}), \quad (2.44)$$

so the left-hand side(LHS) of (2.38) takes the form

$$\begin{aligned} \text{LHS} &= \lambda \iint_{\Omega \times \mathbb{R}^2} f_{\lambda,m} \sqrt{\mu(\vec{v})} \left(|\vec{v}|^2 - \beta_c \right) \left(\sum_{i=1}^2 v_i \partial_i \phi_c \right) \\ &\quad + \epsilon \int_{\partial\Omega \times \mathbb{R}^2} f_{\lambda,m} \sqrt{\mu(\vec{v})} \left(|\vec{v}|^2 - \beta_c \right) \left(\sum_{i=1}^2 v_i \partial_i \phi_c \right) (\vec{n} \cdot \vec{v}) \\ &\quad - \epsilon \iint_{\Omega \times \mathbb{R}^2} f_{\lambda,m} \sqrt{\mu(\vec{v})} \left(|\vec{v}|^2 - \beta_c \right) \left(\sum_{i,j=1}^2 v_i v_j \partial_{ij} \phi_c \right). \end{aligned} \quad (2.45)$$

We decompose

$$(f_{\lambda,m})_\gamma = \mathbf{1}_{\gamma_+} f_{\lambda,m} + \mathbf{1}_{\gamma_-} h \quad \text{on } \gamma, \quad (2.46)$$

$$f_{\lambda,m} = \sqrt{\mu} \left(a + \vec{v} \cdot \vec{b} + \frac{|\vec{v}|^2 - 2}{2} c \right) + (\mathbb{I} - \mathbb{P})[f_{\lambda,m}] \quad \text{in } \Omega \times \mathbb{R}^2. \quad (2.47)$$

Note that the operator \mathbb{P} and $\mathbb{I} - \mathbb{P}$ are defined independent of cut-off parameter m . We will choose β_c such that

$$\int_{\mathbb{R}^2} \sqrt{\mu(\vec{v})} \left(|\vec{v}|^2 - \beta_c \right) v_i^2 d\vec{v} = 0 \quad \text{for } i = 1, 2. \quad (2.48)$$

Since $\mu(\vec{v})$ takes the form

$$\mu(\vec{v}) = C \exp \left(-\frac{|\vec{v}|^2}{2} \right), \quad (2.49)$$

this β_c can always be achieved. Now substitute (2.46) and (2.47) into (2.45). Then based on this choice of β_c and oddness in \vec{v} , there is no $\mathbb{P}[f_{\lambda,m}]$ contribution in the first term and no a contribution in the third term of (2.45). Since \vec{b} contribution and the off-diagonal c contribution in the third term of (2.45) also vanish due to the oddness in \vec{v} , we can simplify (2.45) into

$$\begin{aligned} \text{LHS} &= \lambda \iint_{\Omega \times \mathbb{R}^2} (\mathbb{I} - \mathbb{P})[f_{\lambda,m}] \sqrt{\mu(\vec{v})} \left(|\vec{v}|^2 - \beta_c \right) \left(\sum_{i=1}^2 v_i \partial_i \phi_c \right) \\ &\quad + \epsilon \int_{\partial\Omega \times \mathbb{R}^2} \mathbf{1}_{\gamma_+} f_{\lambda,m} \sqrt{\mu(\vec{v})} \left(|\vec{v}|^2 - \beta_c \right) \left(\sum_{i=1}^2 v_i \partial_i \phi_c \right) (\vec{n} \cdot \vec{v}) \\ &\quad + \epsilon \int_{\partial\Omega \times \mathbb{R}^2} \mathbf{1}_{\gamma_-} h \sqrt{\mu(\vec{v})} \left(|\vec{v}|^2 - \beta_c \right) \left(\sum_{i=1}^2 v_i \partial_i \phi_c \right) (\vec{n} \cdot \vec{v}) \\ &\quad - \epsilon \sum_{i=1}^2 \int_{\mathbb{R}^2} \mu(\vec{v}) |v_i|^2 \left(|\vec{v}|^2 - \beta_c \right) \frac{|\vec{v}|^2 - 2}{2} d\vec{v} \int_{\Omega} c(\vec{x}) \partial_{ii} \phi_c(\vec{x}) d\vec{x} \\ &\quad - \epsilon \iint_{\Omega \times \mathbb{R}^2} (\mathbb{I} - \mathbb{P})[f_{\lambda,m}] \sqrt{\mu(\vec{v})} \left(|\vec{v}|^2 - \beta_c \right) \left(\sum_{i,j=1}^2 v_i v_j \partial_{ij} \phi_c \right). \end{aligned} \quad (2.50)$$

Since

$$\int_{\mathbb{R}^2} \mu(\vec{v}) |v_i|^2 \left(|\vec{v}|^2 - \beta_c \right) \frac{|\vec{v}|^2 - 2}{2} d\vec{v} = C, \quad (2.51)$$

we have

$$\epsilon \left| \int_{\Omega} \Delta_x \phi_c(\vec{x}) c(\vec{x}) d\vec{x} \right| \leq C \|c\|_{L^2} \left(\epsilon |f_{\lambda,m}|_{L^2_+} + (1 + \epsilon + \lambda) \|(\mathbb{I} - \mathbb{P})[f_{\lambda,m}]\|_{L^2} + \|S\|_{L^2} + \epsilon |h|_{L^2_-} \right), \quad (2.52)$$

where we have used the elliptic estimates and the trace estimate: $|\nabla_x \phi_c|_{L^2} \leq C \|\phi_c\|_{H^2} \leq C \|c\|_{L^2}$. Since $-\Delta_x \phi_c = c$, we know

$$\epsilon \|c\|_{L^2}^2 \leq C \|c\|_{L^2} \left(\epsilon |f_{\lambda,m}|_{L^2_+} + (1 + \epsilon + \lambda) \|(\mathbb{I} - \mathbb{P})[f_{\lambda,m}]\|_{L^2} + \|S\|_{L^2} + \epsilon |h|_{L^2_-} \right), \quad (2.53)$$

which further implies

$$\epsilon \|c\|_{L^2} \leq C \epsilon |f_{\lambda,m}|_{L^2_+} + (1 + \epsilon + \lambda) \|(\mathbb{I} - \mathbb{P})[f_{\lambda,m}]\|_{L^2} + \|S\|_{L^2} + \epsilon |h|_{L^2_-}. \quad (2.54)$$

Step 2: Estimates of \vec{b} .

Step 2 - Phase 1: Estimates of $(\partial_{ij} \Delta_x^{-1} b_j) b_i$ for $i, j = 1, 2$.

We choose the test function

$$\psi = \psi_b^{i,j} = \sqrt{\mu(\vec{v})} (v_i^2 - \beta_b) \partial_j \phi_b^j, \quad (2.55)$$

where

$$\begin{cases} -\Delta_x \phi_b^j(\vec{x}) &= b_j(\vec{x}) \text{ in } \Omega, \\ \phi_b^j &= 0 \text{ on } \partial\Omega, \end{cases} \quad (2.56)$$

and β_b is a real number to be determined later. Based on the standard elliptic estimates, we have

$$\|\phi_b^j\|_{H^2} \leq C \|\vec{b}\|_{L^2}. \quad (2.57)$$

With the choice of (2.55), the right-hand side(RHS) of (2.38) is bounded by

$$\text{RHS} \leq C \|\vec{b}\|_{L^2} \left(\|(\mathbb{I} - \mathbb{P})[f_{\lambda,m}]\|_{L^2} + \|S\|_{L^2} \right). \quad (2.58)$$

Hence, the left-hand side(LHS) of (2.38) takes the form

$$\begin{aligned} \text{LHS} &= \lambda \iint_{\Omega \times \mathbb{R}^2} f_{\lambda,m} \sqrt{\mu(\vec{v})} (v_i^2 - \beta_b) \partial_j \phi_b^j \\ &\quad + \epsilon \int_{\partial\Omega \times \mathbb{R}^2} f_{\lambda,m} \sqrt{\mu(\vec{v})} (v_i^2 - \beta_b) \partial_j \phi_b^j (\vec{n} \cdot \vec{v}) \\ &\quad - \epsilon \iint_{\Omega \times \mathbb{R}^2} f_{\lambda,m} \sqrt{\mu(\vec{v})} (v_i^2 - \beta_b) \left(\sum_{i=1}^2 v_i \partial_{ij} \phi_b^j \right). \end{aligned} \quad (2.59)$$

Now substitute (2.46) and (2.47) into (2.59). Then based on the oddness in \vec{v} , there is no \vec{b} contribution in the first term and no a and c contribution in the third term of (2.59). We can simplify (2.59) into

$$\begin{aligned}
\text{LHS} = & \lambda \iint_{\Omega \times \mathbb{R}^2} (\mathbb{I} - \mathbb{P})[f_{\lambda,m}] \sqrt{\mu(\vec{v})} (v_i^2 - \beta_b) \partial_j \phi_b^j \\
& + \lambda \iint_{\Omega \times \mathbb{R}^2} a(\vec{x}) \mu(\vec{v}) (v_i^2 - \beta_b) \partial_j \phi_b^j \\
& + \lambda \iint_{\Omega \times \mathbb{R}^2} c(\vec{x}) \mu(\vec{v}) \frac{|\vec{v}|^2 - 2}{2} (v_i^2 - \beta_b) \partial_j \phi_b^j \\
& + \epsilon \int_{\partial\Omega \times \mathbb{R}^2} \mathbf{1}_{\gamma_+} f_{\lambda,m} \sqrt{\mu(\vec{v})} (v_i^2 - \beta_b) \partial_j \phi_b^j (\vec{n} \cdot \vec{v}) \\
& + \epsilon \int_{\partial\Omega \times \mathbb{R}^2} \mathbf{1}_{\gamma_-} h \sqrt{\mu(\vec{v})} (v_i^2 - \beta_b) \partial_j \phi_b^j (\vec{n} \cdot \vec{v}) \\
& - \epsilon \sum_{l=1}^2 \iint_{\Omega \times \mathbb{R}^2} \mu(\vec{v}) v_l^2 (v_i^2 - \beta_b) \partial_{lj} \phi_b^j(\vec{x}) b_l \\
& - \epsilon \sum_{l=1}^2 \iint_{\Omega \times \mathbb{R}^2} (\mathbb{I} - \mathbb{P})[f_{\lambda,m}] \sqrt{\mu(\vec{v})} (v_i^2 - \beta_b) v_l \partial_{lj} \phi_b^j.
\end{aligned} \tag{2.60}$$

We will choose β_b such that

$$\int_{\mathbb{R}^2} \mu(\vec{v}) (|v_i|^2 - \beta_b) d\vec{v} = 0 \quad \text{for } i = 1, 2. \tag{2.61}$$

Since $\mu(\vec{v})$ takes the form

$$\mu(\vec{v}) = C \exp\left(-\frac{|\vec{v}|^2}{2}\right), \tag{2.62}$$

this β_b can always be achieved. Based on this choice of β_b , we have

$$\lambda \iint_{\Omega \times \mathbb{R}^2} a_{f_{\lambda,m}} \mu(\vec{v}) (v_i^2 - \beta_b) \partial_j \phi_b^j = 0 \tag{2.63}$$

For such β_b and any $i \neq l$, we can directly compute

$$\int_{\mathbb{R}^2} \mu(\vec{v}) (|v_i|^2 - \beta_b) v_l^2 d\vec{v} = 0, \tag{2.64}$$

$$\int_{\mathbb{R}^2} \mu(\vec{v}) (|v_i|^2 - \beta_b) v_i^2 d\vec{v} = C \neq 0. \tag{2.65}$$

Then we deduce

$$\begin{aligned}
& - \epsilon \sum_{l=1}^2 \iint_{\Omega \times \mathbb{R}^2} \mu(\vec{v}) v_l^2 (v_i^2 - \beta_b) \partial_{lj} \phi_b^j(\vec{x}) b_l \\
& = - \epsilon \iint_{\Omega \times \mathbb{R}^2} \mu(\vec{v}) v_i^2 (v_i^2 - \beta_b) \partial_{ij} \phi_b^j(\vec{x}) b_i - \epsilon \sum_{l \neq i} \iint_{\Omega \times \mathbb{R}^2} \mu(\vec{v}) v_l^2 (v_i^2 - \beta_b) \partial_{lj} \phi_b^j(\vec{x}) b_l \\
& = C \int_{\Omega} (\partial_{ij} \Delta_x^{-1} b_j) b_i,
\end{aligned} \tag{2.66}$$

and

$$\lambda \iint_{\Omega \times \mathbb{R}^2} c(\vec{x}) \mu(\vec{v}) \frac{|\vec{v}|^2 - 2}{2} (v_i^2 - \beta_b) \partial_j \phi_b^j = \lambda \iint_{\Omega \times \mathbb{R}^2} c(\vec{x}) \mu(\vec{v}) \frac{v_i^2 - 2}{2} (v_i^2 - \beta_b) \partial_j \phi_b^j. \tag{2.67}$$

Hence, by (2.54), we may estimate

$$\begin{aligned}
 & \epsilon \left| \int_{\Omega} (\partial_{ij} \Delta_x^{-1} b_j) b_i \right| \\
 & \leq C \left\| \vec{b} \right\|_{L^2} \left(\epsilon |f_{\lambda, m}|_{L^2_+} + (1 + \epsilon + \lambda) \|(\mathbb{I} - \mathbb{P})[f_{\lambda, m}]\|_{L^2} + \|S\|_{L^2} + \epsilon |h|_{L^2_-} + \lambda \|c\|_{L^2} \right) \\
 & \leq C \left\| \vec{b} \right\|_{L^2} \left((\epsilon + \lambda) |f_{\lambda, m}|_{L^2_+} + (1 + \epsilon + \lambda) \|(\mathbb{I} - \mathbb{P})[f_{\lambda, m}]\|_{L^2} + (1 + \lambda) \|S\|_{L^2} + (\epsilon + \lambda) |h|_{L^2_-} \right).
 \end{aligned} \tag{2.68}$$

Step 2 - Phase 2: Estimates of $(\partial_{jj} \Delta_x^{-1} b_i) b_i$ for $i \neq j$.

We choose the test function

$$\psi = \sqrt{\mu(\vec{v})} |\vec{v}|^2 v_i v_j \partial_j \phi_b^i \quad i \neq j. \tag{2.69}$$

The right-hand side(RHS) of (2.38) is still bounded by

$$\text{RHS} \leq C \left\| \vec{b} \right\|_{L^2} \left(\|(\mathbb{I} - \mathbb{P})[f_{\lambda, m}]\|_{L^2} + \|S\|_{L^2} \right). \tag{2.70}$$

Hence, the left-hand side(LHS) of (2.38) takes the form

$$\begin{aligned}
 \text{LHS} &= \lambda \iint_{\Omega \times \mathbb{R}^2} f_{\lambda, m} \sqrt{\mu(\vec{v})} |\vec{v}|^2 v_i v_j \partial_j \phi_b^i \\
 &+ \epsilon \int_{\partial \Omega \times \mathbb{R}^2} f_{\lambda, m} \sqrt{\mu(\vec{v})} |\vec{v}|^2 v_i v_j \partial_j \phi_b^i (\vec{n} \cdot \vec{v}) \\
 &- \epsilon \iint_{\Omega \times \mathbb{R}^2} f_{\lambda, m} \sqrt{\mu(\vec{v})} |\vec{v}|^2 v_i v_j \left(\sum_{l=1}^2 v_l \partial_{lj} \phi_b^i \right).
 \end{aligned} \tag{2.71}$$

Now substitute (2.46) and (2.47) into (2.71). Then based on the oddness in \vec{v} , there is no $\mathbb{P}[f_{\lambda, m}]$ contribution in the first term and no a and c contribution in the third term of (2.71). We can simplify (2.71) into

$$\begin{aligned}
 \text{LHS} &= \lambda \iint_{\Omega \times \mathbb{R}^2} (\mathbb{I} - \mathbb{P})[f_{\lambda, m}] \sqrt{\mu(\vec{v})} |\vec{v}|^2 v_i v_j \partial_j \phi_b^i \\
 &+ \epsilon \int_{\partial \Omega \times \mathbb{R}^2} \mathbf{1}_{\gamma_+} f_{\lambda, m} \sqrt{\mu(\vec{v})} |\vec{v}|^2 v_i v_j \partial_j \phi_b^i (\vec{n} \cdot \vec{v}) \\
 &+ \epsilon \int_{\partial \Omega \times \mathbb{R}^2} \mathbf{1}_{\gamma_-} h \sqrt{\mu(\vec{v})} |\vec{v}|^2 v_i v_j \partial_j \phi_b^i (\vec{n} \cdot \vec{v}) \\
 &- \epsilon \iint_{\Omega \times \mathbb{R}^2} \mu(\vec{v}) |\vec{v}|^2 v_i^2 v_j^2 (\partial_{ij} \phi_b^i(\vec{x}) b_j + \partial_{jj} \phi_b^i(\vec{x}) b_i) \\
 &- \epsilon \sum_{l=1}^2 \iint_{\Omega \times \mathbb{R}^2} (\mathbb{I} - \mathbb{P})[f_{\lambda, m}] \sqrt{\mu(\vec{v})} |\vec{v}|^2 v_i v_j v_l \partial_{lj} \phi_b^i.
 \end{aligned} \tag{2.72}$$

Then we deduce

$$- \epsilon \iint_{\Omega \times \mathbb{R}^2} \mu(\vec{v}) |\vec{v}|^2 v_i^2 v_j^2 (\partial_{ij} \phi_b^i(\vec{x}) b_j + \partial_{jj} \phi_b^i(\vec{x}) b_i) = C \left(\int_{\Omega} (\partial_{ij} \Delta_x^{-1} b_i) b_j + \int_{\Omega} (\partial_{jj} \Delta_x^{-1} b_i) b_i \right). \tag{2.73}$$

Hence, we may estimate for $i \neq j$,

$$\begin{aligned}
 \left| \int_{\Omega} (\partial_{jj} \Delta_x^{-1} b_i) b_i \right| &\leq C \left\| \vec{b} \right\|_{L^2} \left(\epsilon |f_{\lambda, m}|_{L^2_+} + (1 + \epsilon + \lambda) \|(\mathbb{I} - \mathbb{P})[f_{\lambda, m}]\|_{L^2} \right. \\
 &\quad \left. + \|S\|_{L^2} + \epsilon |h|_{L^2_-} \right) + C \epsilon \left| \int_{\Omega} (\partial_{ij} \Delta_x^{-1} b_i) b_j \right|.
 \end{aligned} \tag{2.74}$$

Moreover, by (2.68), for $i = j = 1, 2$,

$$\begin{aligned} \epsilon \left| \int_{\Omega} (\partial_{jj} \Delta_x^{-1} b_j) b_j \right| &\leq C \left\| \vec{b} \right\|_{L^2} \left((\epsilon + \lambda) |f_{\lambda, m}|_{L^2_+} + (1 + \epsilon + \lambda) \|(\mathbb{I} - \mathbb{P})[f_{\lambda, m}]\|_{L^2} \right. \\ &\quad \left. + (1 + \lambda) \|S\|_{L^2} + (\epsilon + \lambda) |h|_{L^2_-} \right). \end{aligned} \quad (2.75)$$

Step 2 - Phase 3: Synthesis.

Summarizing (2.74) and (2.75), we may sum up over $j = 1, 2$ to obtain, for any $i = 1, 2$,

$$\begin{aligned} \epsilon \|b_i\|_{L^2}^2 &\leq C \left\| \vec{b} \right\|_{L^2} \left((\epsilon + \lambda) |f_{\lambda, m}|_{L^2_+} + (1 + \epsilon + \lambda) \|(\mathbb{I} - \mathbb{P})[f_{\lambda, m}]\|_{L^2} \right. \\ &\quad \left. + (1 + \lambda) \|S\|_{L^2} + (\epsilon + \lambda) |h|_{L^2_-} \right), \end{aligned} \quad (2.76)$$

which further implies

$$\epsilon \left\| \vec{b} \right\|_{L^2} \leq C \left((\epsilon + \lambda) |f_{\lambda, m}|_{L^2_+} + (1 + \epsilon + \lambda) \|(\mathbb{I} - \mathbb{P})[f_{\lambda, m}]\|_{L^2} + (1 + \lambda) \|S\|_{L^2} + (\epsilon + \lambda) |h|_{L^2_-} \right). \quad (2.77)$$

Step 3: Estimates of a .

We choose the test function

$$\psi = \psi_a = \sqrt{\mu(\vec{v})} \left(|\vec{v}|^2 - \beta_a \right) (\vec{v} \cdot \nabla_x \phi_a(\vec{x})), \quad (2.78)$$

where

$$\begin{cases} -\Delta_x \phi_a(\vec{x}) &= a(\vec{x}) \text{ in } \Omega, \\ \phi_a &= 0 \text{ on } \partial\Omega, \end{cases} \quad (2.79)$$

and β_a is a real number to be determined later. Based on the standard elliptic estimates, we have

$$\|\phi_a\|_{H^2} \leq C \|a\|_{L^2}. \quad (2.80)$$

With the choice of (2.78), the right-hand side(RHS) of (2.38) is bounded by

$$\text{RHS} \leq C \|a\|_{L^2} \left(\|(\mathbb{I} - \mathbb{P})[f_{\lambda, m}]\|_{L^2} + \|S\|_{L^2} \right). \quad (2.81)$$

We have

$$\vec{v} \cdot \nabla_x \psi_a = \sqrt{\mu(\vec{v})} \sum_{i,j=1}^2 \left(|\vec{v}|^2 - \beta_a \right) v_i v_j \partial_{ij} \phi_a(\vec{x}), \quad (2.82)$$

so the left-hand side(LHS) of (2.38) takes the form

$$\begin{aligned} \text{LHS} &= \lambda \iint_{\Omega \times \mathbb{R}^2} f_{\lambda, m} \sqrt{\mu(\vec{v})} \left(|\vec{v}|^2 - \beta_a \right) \left(\sum_{i=1}^2 v_i \partial_i \phi_a \right) \\ &\quad + \epsilon \int_{\partial\Omega \times \mathbb{R}^2} f_{\lambda, m} \sqrt{\mu(\vec{v})} \left(|\vec{v}|^2 - \beta_a \right) \left(\sum_{i=1}^2 v_i \partial_i \phi_a \right) (\vec{n} \cdot \vec{v}) \\ &\quad - \epsilon \iint_{\Omega \times \mathbb{R}^2} f_{\lambda, m} \sqrt{\mu(\vec{v})} \left(|\vec{v}|^2 - \beta_a \right) \left(\sum_{i,j=1}^2 v_i v_j \partial_{ij} \phi_a \right). \end{aligned} \quad (2.83)$$

We will choose β_a such that

$$\int_{\mathbb{R}^2} \sqrt{\mu(\vec{v})} \left(|\vec{v}|^2 - \beta_a \right) \frac{|\vec{v}|^2 - 2}{2} v_i^2 d\vec{v} = 0 \text{ for } i = 1, 2. \quad (2.84)$$

Since

$$\int_{\mathbb{R}^2} \sqrt{\mu(\vec{v})} \frac{|\vec{v}|^2 - 2}{2} v_i^2 d\vec{v} \neq 0, \quad (2.85)$$

this β_a can always be achieved. Now substitute (2.46) and (2.47) into (2.83). Then based on this choice of β_a and oddness in \vec{v} , there is no a and c contribution in the first term, and no \vec{b} and c contribution in the third term of (2.83). Since \vec{b} contribution and the off-diagonal c contribution in the third term of (2.83) also vanish due to the oddness in \vec{v} , we can simplify (2.83) into

$$\begin{aligned} \text{LHS} = & \lambda \iint_{\Omega \times \mathbb{R}^2} (\mathbb{I} - \mathbb{P})[f_{\lambda,m}] \sqrt{\mu(\vec{v})} \left(|\vec{v}|^2 - \beta_a \right) \left(\sum_{i=1}^2 v_i \partial_i \phi_a \right) \\ & + \lambda \iint_{\Omega \times \mathbb{R}^2} \mu(\vec{v}) \left(|\vec{v}|^2 - \beta_a \right) \left(\sum_{i=1}^2 b_i v_i^2 \partial_i \phi_a \right) \\ & + \epsilon \int_{\partial\Omega \times \mathbb{R}^2} \mathbf{1}_{\gamma_+} f_{\lambda,m} \sqrt{\mu(\vec{v})} \left(|\vec{v}|^2 - \beta_a \right) \left(\sum_{i=1}^2 v_i \partial_i \phi_a \right) (\vec{n} \cdot \vec{v}) \\ & + \epsilon \int_{\partial\Omega \times \mathbb{R}^2} \mathbf{1}_{\gamma_-} h \sqrt{\mu(\vec{v})} \left(|\vec{v}|^2 - \beta_a \right) \left(\sum_{i=1}^2 v_i \partial_i \phi_a \right) (\vec{n} \cdot \vec{v}) \\ & - \sum_{i=1}^2 \epsilon \int_{\mathbb{R}^2} \mu(\vec{v}) |v_i|^2 \left(|\vec{v}|^2 - \beta_a \right) d\vec{v} \int_{\Omega} a(\vec{x}) \partial_{ii} \phi_a(\vec{x}) d\vec{x} \\ & - \epsilon \iint_{\Omega \times \mathbb{R}^2} (\mathbb{I} - \mathbb{P})[f_{\lambda,m}] \sqrt{\mu(\vec{v})} \left(|\vec{v}|^2 - \beta_a \right) \left(\sum_{i,j=1}^2 v_i v_j \partial_{ij} \phi_a \right). \end{aligned} \quad (2.86)$$

Since

$$\int_{\mathbb{R}^2} \sqrt{\mu(\vec{v})} |v_i|^2 \left(|\vec{v}|^2 - \beta_a \right) d\vec{v} = C, \quad (2.87)$$

we have

$$\begin{aligned} & - \epsilon \int_{\Omega} \Delta_x \phi_a(\vec{x}) a(\vec{x}) d\vec{x} \\ & \leq C \|a\|_{L^2} \left(\epsilon \|f_{\lambda,m}\|_{L^2_+} + (1 + \epsilon + \lambda) \|(\mathbb{I} - \mathbb{P})[f_{\lambda,m}]\|_{L^2} + \|S\|_{L^2} + \epsilon \|h\|_{L^2_-} + \lambda \|\vec{b}\|_{L^2} \right). \end{aligned} \quad (2.88)$$

Since $-\Delta_x \phi_a = a$, by (2.77), we know

$$\begin{aligned} \epsilon \|a\|_{L^2}^2 & \leq C \|a\|_{L^2} \left((\epsilon + \lambda) \|f_{\lambda,m}\|_{L^2_+} + (1 + \epsilon + \lambda) \|(\mathbb{I} - \mathbb{P})[f_{\lambda,m}]\|_{L^2} \right. \\ & \quad \left. + (1 + \lambda) \|S\|_{L^2} + (\epsilon + \lambda) \|h\|_{L^2_-} \right), \end{aligned} \quad (2.89)$$

which further implies

$$\epsilon \|a\|_{L^2} \leq C \left((\epsilon + \lambda) \|f_{\lambda,m}\|_{L^2_+} + (1 + \epsilon + \lambda) \|(\mathbb{I} - \mathbb{P})[f_{\lambda,m}]\|_{L^2} + (1 + \lambda) \|S\|_{L^2} + (\epsilon + \lambda) \|h\|_{L^2_-} \right). \quad (2.90)$$

Step 4: Synthesis.

Collecting (2.54), (2.77) and (2.90), we deduce

$$\begin{aligned} & \epsilon \|\mathbb{P}[f_{\lambda,m}]\|_{L^2} \\ & \leq C \left((\epsilon + \lambda) \|f_{\lambda,m}\|_{L^2_+} + (1 + \epsilon + \lambda) \|(\mathbb{I} - \mathbb{P})[f_{\lambda,m}]\|_{L^2} + (1 + \lambda) \|S\|_{L^2} + (\epsilon + \lambda) \|h\|_{L^2_-} \right). \end{aligned} \quad (2.91)$$

This completes our proof. \square

Theorem 2.5. *There exists a unique solution $f \in L^2(\Omega \times \mathbb{R}^2)$ to the equation (2.1) that satisfies the estimate*

$$\|f\|_{L^2} + \frac{1}{\epsilon^{1/2}} |f|_{L^2_+} \leq C \left(\frac{1}{\epsilon^2} \|(\mathbb{I} - \mathbb{P})[S]\|_{L^2} + \frac{1}{\epsilon} \|\mathbb{P}[S]\|_{L^2} + \frac{1}{\epsilon^{1/2}} |h|_{L^2_-} \right). \quad (2.92)$$

Proof. We square on both sides of (2.37) to obtain

$$\epsilon^2 \|\mathbb{P}[f_{\lambda,m}]\|_{L^2}^2 \leq C \left(\epsilon^2 |f_{\lambda,m}|_{L^2_+}^2 + \|(\mathbb{I} - \mathbb{P})[f_{\lambda,m}]\|_{L^2}^2 + \|S\|_{L^2}^2 + \epsilon^2 |h|_{L^2_-}^2 \right). \quad (2.93)$$

On the other hand, by Green's identity, multiplying $f_{\lambda,m}$ on both sides of (2.24) implies

$$\lambda \|f_{\lambda,m}\|_{L^2}^2 + \langle \mathcal{L}_m f_{\lambda,m}, f_{\lambda,m} \rangle + \frac{1}{2} \epsilon |f_{\lambda,m}|_{L^2_+}^2 = \frac{1}{2} \epsilon |h|_{L^2_-}^2 + \iint_{\Omega \times \mathbb{R}^2} f_{\lambda,m} S. \quad (2.94)$$

We deduce from the spectral gap of \mathcal{L}_m ,

$$\lambda \|f_{\lambda,m}\|_{L^2}^2 + \|(\mathbb{I} - \mathbb{P})[f_{\lambda,m}]\|_{L^2}^2 + \frac{\epsilon}{2} |f_{\lambda,m}|_{L^2_+}^2 \leq \epsilon |h|_{L^2_-}^2 + \iint_{\Omega \times \mathbb{R}^2} f_{\lambda,m} S. \quad (2.95)$$

Multiplying a small constant on both sides of (2.93) and adding to (2.95), we obtain

$$\epsilon^2 \|\mathbb{P}[f_{\lambda,m}]\|_{L^2}^2 + \|(\mathbb{I} - \mathbb{P})[f_{\lambda,m}]\|_{L^2}^2 + \epsilon |f_{\lambda,m}|_{L^2_+}^2 \leq C \left(\epsilon |h|_{L^2_-}^2 + \iint_{\Omega \times \mathbb{R}^2} f_{\lambda,m} S + \|S\|_{L^2}^2 \right). \quad (2.96)$$

Since

$$\begin{aligned} \iint_{\Omega \times \mathbb{R}^2} f_{\lambda,m} S &= \iint_{\Omega \times \mathbb{R}^2} f_{\lambda,m} \mathbb{P}[S] + \iint_{\Omega \times \mathbb{R}^2} f_{\lambda,m} (\mathbb{I} - \mathbb{P})[S] \\ &= \iint_{\Omega \times \mathbb{R}^2} (\mathbb{I} - \mathbb{P})[f_{\lambda,m}] \mathbb{P}[S] + \iint_{\Omega \times \mathbb{R}^2} \mathbb{P}[f_{\lambda,m}] (\mathbb{I} - \mathbb{P})[S] \\ &\leq \left(C \|(\mathbb{I} - \mathbb{P})[f_{\lambda,m}]\|_{L^2}^2 + \frac{1}{4C} \|\mathbb{P}[S]\|_{L^2}^2 \right) + \left(C \epsilon^2 \|\mathbb{P}[f_{\lambda,m}]\|_{L^2}^2 + \frac{1}{4C\epsilon^2} \|(\mathbb{I} - \mathbb{P})[S]\|_{L^2}^2 \right), \end{aligned} \quad (2.97)$$

for C sufficiently small, we have

$$\epsilon^2 \|\mathbb{P}[f_{\lambda,m}]\|_{L^2}^2 + \|(\mathbb{I} - \mathbb{P})[f_{\lambda,m}]\|_{L^2}^2 + \epsilon |f_{\lambda,m}|_{L^2_+}^2 \leq C \left(\frac{1}{\epsilon^2} \|(\mathbb{I} - \mathbb{P})[S]\|_{L^2}^2 + \|\mathbb{P}[S]\|_{L^2}^2 + \epsilon |h|_{L^2_-}^2 \right). \quad (2.98)$$

Hence, we deduce

$$\|f_{\lambda,m}\|_{L^2} \leq C \left(\frac{1}{\epsilon^2} \|(\mathbb{I} - \mathbb{P})[S]\|_{L^2} + \frac{1}{\epsilon} \|\mathbb{P}[S]\|_{L^2} + \frac{1}{\epsilon^{1/2}} |h|_{L^2_-} \right). \quad (2.99)$$

Then based on (2.96), we have

$$|f_{\lambda,m}|_{L^2_+} \leq C \left(\frac{1}{\epsilon^{3/2}} \|(\mathbb{I} - \mathbb{P})[S]\|_{L^2} + \frac{1}{\epsilon^{1/2}} \|\mathbb{P}[S]\|_{L^2} + |h|_{L^2_-} \right). \quad (2.100)$$

This is a uniform estimate in λ , so we can obtain a weak solution $f_{\lambda,m} \rightarrow f_m$ with the same estimate (2.99) and (2.100). Moreover, we have

$$\begin{cases} \lambda(f_{\lambda,m} - f_m) + \epsilon \vec{v} \cdot \nabla_x (f_{\lambda,m} - f_m) + \mathcal{L}_m[f_{\lambda,m} - f_m] &= \lambda f_m, \\ (f_{\lambda,m} - f_m)(\vec{x}_0, \vec{v}) &= 0. \end{cases} \quad (2.101)$$

Then we have the estimate

$$\|f_{\lambda,m} - f_m\|_{L^2} \leq C \left(\frac{\lambda}{\epsilon^2} \|(\mathbb{I} - \mathbb{P})[f_m]\|_{L^2} + \frac{\lambda}{\epsilon} \|\mathbb{P}[f_m]\|_{L^2} \right). \quad (2.102)$$

Hence, $f_{\lambda,m} \rightarrow f_m$ strongly in $L^2(\Omega \times \mathbb{R}^2)$ as $\lambda \rightarrow 0$. Then we can take the limit $f_m \rightarrow f$ as $m \rightarrow \infty$. By a diagonal process, there exists a unique weak solution such that $f_m \rightarrow f$ weakly in $L^2(\Omega \times \mathbb{R}^2)$. Then the weak lower semi-continuity implies f satisfies the same estimate (2.99). \square

2.3. L^∞ Estimates of Linearized Stationary Boltzmann Equation. The characteristics $(X(s), V(s))$ of the equation (2.1) which goes through (\vec{x}, \vec{v}) is defined by

$$\begin{cases} (X(0), V(0)) &= (\vec{x}, \vec{v}) \\ \frac{dX(s)}{ds} &= \epsilon V(s), \\ \frac{dV(s)}{ds} &= 0. \end{cases} \quad (2.103)$$

which implies

$$\begin{cases} X(s) &= \vec{x} + \epsilon s \vec{v} \\ V(s) &= \vec{v} \end{cases} \quad (2.104)$$

Define the backward exit time and exit position as

$$t_b(\vec{x}, \vec{v}) = \inf\{t > 0 : \vec{x} - \epsilon t \vec{v} \notin \Omega\}, \quad (2.105)$$

$$x_b(\vec{x}, \vec{v}) = \vec{x} - \epsilon t_b(\vec{x}, \vec{v}) \vec{v} \notin \Omega. \quad (2.106)$$

We define a weight function

$$w(\vec{v}) = w_{\vartheta, \zeta}(\vec{v}) = \langle \vec{v} \rangle^\vartheta e^{\zeta |\vec{v}|^2}, \quad (2.107)$$

and

$$\tilde{w}(\vec{v}) = \frac{1}{\sqrt{\mu(\vec{v})w(\vec{v})}} = \sqrt{2\pi} \frac{e^{\left(\frac{1}{4}-\zeta\right)|\vec{v}|^2}}{\langle \vec{v} \rangle^\vartheta}. \quad (2.108)$$

Lemma 2.6. *We have*

$$|k(\vec{v}, \vec{v}')| \leq C \left(|\vec{v} - \vec{v}'| + \left| \frac{1}{\vec{v} - \vec{v}'} \right| \right) \exp \left(-\frac{1}{8} |\vec{v} - \vec{v}'|^2 - \frac{1}{8} \frac{|\vec{v}|^2 - |\vec{v}'|^2}{|\vec{v} - \vec{v}'|^2} \right). \quad (2.109)$$

Let $0 \leq \zeta \leq 1/4$. Then there exists $0 \leq C_1(\zeta) < 1$ and $C_2(\zeta) > 0$ such that for $0 \leq \delta \leq C_1(\zeta)$,

$$(2.110)$$

$$\int_{\mathbb{R}^2} \left(|\vec{v} - \vec{v}'| + \left| \frac{1}{\vec{v} - \vec{v}'} \right| \right) \exp \left(-\frac{1-\delta}{8} |\vec{v} - \vec{v}'|^2 - \frac{1-\delta}{8} \frac{|\vec{v}|^2 - |\vec{v}'|^2}{|\vec{v} - \vec{v}'|^2} \right) \frac{e^{\zeta |\vec{v}|^2}}{e^{\zeta |\vec{v}'|^2}} d\vec{v}' \leq \frac{C_2(\zeta)}{1 + |\vec{v}|}.$$

Proof. See [14, Lemma 3]. \square

Theorem 2.7. *There exists a unique solution $f \in L^\infty(\Omega \times \mathbb{R}^2)$ to the equation (2.1) that satisfies the estimate for $\vartheta > 2$ and $0 \leq \zeta \leq 1/4$,*

$$\begin{aligned} & \left\| \langle \vec{v} \rangle^\vartheta e^{\zeta |\vec{v}|^2} f \right\|_{L^\infty} + \left| \langle \vec{v} \rangle^\vartheta e^{\zeta |\vec{v}|^2} f \right|_{L^\infty_+} \\ & \leq C \left(\frac{1}{\epsilon} \|f\|_{L^2} + \left\| \langle \vec{v} \rangle^\vartheta e^{\zeta |\vec{v}|^2} S \right\|_{L^\infty} + \left| \langle \vec{v} \rangle^\vartheta e^{\zeta |\vec{v}|^2} h \right|_{L^\infty_-} \right) \\ & \leq C \left(\frac{1}{\epsilon^3} \|(\mathbb{I} - \mathbb{P})[S]\|_{L^2} + \frac{1}{\epsilon^2} \|\mathbb{P}[S]\|_{L^2} + \frac{1}{\epsilon^{3/2}} \|h\|_{L^2} + \left\| \langle \vec{v} \rangle^\vartheta e^{\zeta |\vec{v}|^2} S \right\|_{L^\infty} + \left| \langle \vec{v} \rangle^\vartheta e^{\zeta |\vec{v}|^2} h \right|_{L^\infty_-} \right). \end{aligned} \quad (2.111)$$

Proof. The existence and uniqueness follow from Theorem 2.5, so we focus on the estimate. We divide the proof into several steps:

Step 1: Mild formulation.

Denote

$$g = wf, \quad (2.112)$$

$$K_w[g] = wK \left[\frac{g}{w} \right] = \int_{\mathbb{R}^2} k_w(\vec{v}, \vec{v}') g(\vec{v}') d\vec{v}'. \quad (2.113)$$

Since $\mathcal{L} = \nu - K$, we can rewrite the equation (2.1) along the characteristics by Duhamel's principle as

$$\begin{aligned}
g(\vec{x}, \vec{v}) &= w(\vec{v})h(\vec{x} - \epsilon t_b \vec{v}, \vec{v})e^{-\nu(\vec{v})t_b} + \int_0^{t_b} wS(\vec{x} - \epsilon(t_b - s)\vec{v}, \vec{v})e^{-\nu(\vec{v})(t_b - s)}ds \\
&\quad + \int_0^{t_b} K_w[g(\vec{x} - \epsilon(t_b - s)\vec{v}, \vec{v})]e^{-\nu(\vec{v})(t_b - s)}ds \\
&= w(\vec{v})h(\vec{x} - \epsilon t_b \vec{v}, \vec{v})e^{-\nu(\vec{v})t_b} + \int_0^{t_b} wS(\vec{x} - \epsilon(t_b - s)\vec{v}, \vec{v})e^{-\nu(\vec{v})(t_b - s)}ds \\
&\quad + \int_0^{t_b} \left(\int_{\mathbb{R}^2} k_w(\vec{v}, \vec{v}_t)g(\vec{x} - \epsilon(t_b - s)\vec{v}, \vec{v}_t)d\vec{v}_t \right) e^{-\nu(\vec{v})(t_b - s)}ds,
\end{aligned} \tag{2.114}$$

where $\vec{v}_t \in \mathbb{R}^2$ is a dummy variable. We may apply Duhamel's principle again to $g(\vec{x} - \epsilon(t_b - s)\vec{v}, \vec{v}_t)$ to obtain

(2.115)

$$\begin{aligned}
g(\vec{x}, \vec{v}) &= w(\vec{v})h(\vec{x} - \epsilon t_b \vec{v}, \vec{v})e^{-\nu(\vec{v})t_b} + \int_0^{t_b} wS(\vec{x} - \epsilon(t_b - s)\vec{v}, \vec{v})e^{-\nu(\vec{v})(t_b - s)}ds \\
&\quad + \int_0^{t_b} \left(\int_{\mathbb{R}^2} k_w(\vec{v}, \vec{v}_t)w(\vec{v}_t)h(\vec{x} - \epsilon t_b \vec{v} - \epsilon s_b \vec{v}_t, \vec{v}_t)e^{-\nu(\vec{v}_t)s_b}d\vec{v}_t \right) e^{-\nu(\vec{v})(t_b - s)}ds \\
&\quad + \int_0^{t_b} \int_{\mathbb{R}^2} k_w(\vec{v}, \vec{v}_t) \left(\int_0^{s_b} wS(\vec{x} - \epsilon(t_b - s)\vec{v} - \epsilon(s_b - r)\vec{v}_t, \vec{v}_t)e^{-\nu(\vec{v}_t)(s_b - r)}dr \right) d\vec{v}_t e^{-\nu(\vec{v})(t_b - s)}ds \\
&\quad + \int_0^{t_b} \int_{\mathbb{R}^2} k_w(\vec{v}, \vec{v}_t) \left(\int_0^{s_b} \int_{\mathbb{R}^2} k_w(\vec{v}_t, \vec{v}_s)g(\vec{x} - \epsilon(t_b - s)\vec{v} - \epsilon(s_b - r)\vec{v}_t, \vec{v}_s)d\vec{v}_s e^{-\nu(\vec{v}_t)(s_b - r)}dr \right) d\vec{v}_t e^{-\nu(\vec{v})(t_b - s)}ds,
\end{aligned}$$

where

$$s_b(\vec{x}, \vec{v}, \vec{v}_t, s) = \inf\{r > 0 : \vec{x} - \epsilon(t_b - s)\vec{v} - \epsilon r \vec{v}_t \notin \Omega\}, \tag{2.116}$$

and $\vec{v}_s \in \mathbb{R}^2$ is another dummy variable. We need to estimate each term in (2.115).

Step 2: Estimates of source terms and boundary terms.

We can directly obtain

$$\left| w(\vec{v})h(\vec{x} - \epsilon t_b \vec{v}, \vec{v})e^{-\nu(\vec{v})t_b} \right| \leq C |wh|_{L^\infty_-}, \tag{2.117}$$

$$\left| \int_0^{t_b} wS(\vec{x} - \epsilon(t_b - s)\vec{v}, \vec{v})e^{-\nu(\vec{v})(t_b - s)}ds \right| \leq C \|wS\|_{L^\infty}, \tag{2.118}$$

$$\left| \int_0^{t_b} \left(\int_{\mathbb{R}^2} k_w(\vec{v}, \vec{v}_t)w(\vec{v}_t)h(\vec{x} - \epsilon t_b \vec{v} - \epsilon s_b \vec{v}_t, \vec{v}_t)e^{-\nu(\vec{v}_t)s_b}d\vec{v}_t \right) e^{-\nu(\vec{v})(t_b - s)}ds \right| \tag{2.119}$$

$$\leq |wh|_{L^\infty_-} \left| \int_0^{t_b} \left(\int_{\mathbb{R}^2} k_w(\vec{v}, \vec{v}_t)e^{-\nu(\vec{v}_t)s_b}d\vec{v}_t \right) e^{-\nu(\vec{v})(t_b - s)}ds \right| \leq C |wh|_{L^\infty_-},$$

(2.120)

$$\begin{aligned}
&\left| \int_0^{t_b} \int_{\mathbb{R}^2} k_w(\vec{v}, \vec{v}_t) \left(\int_0^{s_b} wS(\vec{x} - \epsilon(t_b - s)\vec{v} - \epsilon(s_b - r)\vec{v}_t, \vec{v}_t)e^{-\nu(\vec{v}_t)(s_b - r)}dr \right) d\vec{v}_t e^{-\nu(\vec{v})(t_b - s)}ds \right| \\
&\leq \|wS\|_{L^\infty} \left| \int_0^{t_b} \int_{\mathbb{R}^2} k_w(\vec{v}, \vec{v}_t) \left(\int_0^{s_b} e^{-\nu(\vec{v}_t)(s_b - r)}dr \right) d\vec{v}_t e^{-\nu(\vec{v})(t_b - s)}ds \right| \leq C \|wS\|_{L^\infty}.
\end{aligned}$$

Step 3: Estimates of K_w terms.

We consider the last and most complicated term I in (2.115) defined as

$$\int_0^{t_b} \int_{\mathbb{R}^2} k_w(\vec{v}, \vec{v}_t) \left(\int_0^{s_b} \int_{\mathbb{R}^2} k_w(\vec{v}_t, \vec{v}_s) g(\vec{x} - \epsilon(t_b - s)\vec{v} - \epsilon(s_b - r)\vec{v}_t, \vec{v}_s) d\vec{v}_s e^{-\nu(\vec{v}_t)(s_b - r)} dr \right) d\vec{v}_t e^{-\nu(\vec{v})(t_b - s)} ds. \quad (2.121)$$

We can divide it into four cases:

$$I = I_1 + I_2 + I_3 + I_4. \quad (2.122)$$

Case I: $|\vec{v}| \geq N$.

Based on Lemma 2.6 with $\delta = 0$, we have

$$\left| \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} k_w(\vec{v})(\vec{v}, \vec{v}_t) k_w(\vec{v}_t)(\vec{v}_t, \vec{v}_s) d\vec{v}_s d\vec{v}_t \right| \leq \frac{C}{1 + |\vec{v}|} \leq \frac{C}{N}. \quad (2.123)$$

Hence, by Fubini's Theorem, we get

$$I_1 \leq \frac{C}{N} \|g\|_{L^\infty}. \quad (2.124)$$

Case II: $|\vec{v}| \leq N$, $|\vec{v}_t| \geq 2N$, or $|\vec{v}_t| \leq 2N$, $|\vec{v}_s| \geq 3N$.

Notice this implies either $|\vec{v}_t - \vec{v}| \geq N$ or $|\vec{v}_t - \vec{v}_s| \geq N$. Hence, either of the following is valid correspondingly:

$$|k_w(\vec{v})(\vec{v}, \vec{v}_t)| \leq e^{-\frac{\delta}{8}N^2} \left| k_w(\vec{v})(\vec{v}, \vec{v}_t) e^{\frac{\delta}{8}|\vec{v} - \vec{v}_t|^2} \right|, \quad (2.125)$$

$$|k_w(\vec{v}_t)(\vec{v}_t, \vec{v}_s)| \leq e^{-\frac{\delta}{8}N^2} \left| k_w(\vec{v}_t)(\vec{v}_t, \vec{v}_s) e^{\frac{\delta}{8}|\vec{v}_t - \vec{v}_s|^2} \right|. \quad (2.126)$$

Then based on Lemma 2.6,

$$\int_{\mathbb{R}^2} \left| k_w(\vec{v})(\vec{v}, \vec{v}_t) e^{\frac{\delta}{8}|\vec{v} - \vec{v}_t|^2} \right| d\vec{v}_t < \infty, \quad (2.127)$$

$$\int_{\mathbb{R}^2} \left| k_w(\vec{v}_t)(\vec{v}_t, \vec{v}_s) e^{\frac{\delta}{8}|\vec{v}_t - \vec{v}_s|^2} \right| d\vec{v}_s < \infty. \quad (2.128)$$

Hence, we have

$$I_2 \leq C e^{-\frac{\delta}{8}N^2} \|g\|_{L^\infty}. \quad (2.129)$$

Case III: $s - r \leq \delta$ and $|\vec{v}| \leq N$, $|\vec{v}_t| \leq 2N$, $|\vec{v}_s| \leq 3N$.

In this case, when $0 < \delta < 1$ is sufficiently small, since the integral in r is restricted to this short interval, we have

$$I_3 \leq C\delta \|g\|_{L^\infty}. \quad (2.130)$$

Case IV: $s - r \geq \delta$ and $|\vec{v}| \leq N$, $|\vec{v}_t| \leq 2N$, $|\vec{v}_s| \leq 3N$.

Since $k_w(\vec{v})(\vec{v}, \vec{v}_t)$ has possible integrable singularity of $1/|\vec{v} - \vec{v}_t|$, we can introduce $k_N(\vec{v}, \vec{v}_t)$ smooth with compact support such that

$$\sup_{|p| \leq 3N} \int_{|\vec{v}_t| \leq 3N} |k_N(p, \vec{v}_t) - k_w(p)(p, \vec{v}_t)| d\vec{v}_t \leq \frac{1}{N}. \quad (2.131)$$

Then we can split

$$\begin{aligned} k_w(\vec{v})(\vec{v}, \vec{v}_t) k_w(\vec{v}_t)(\vec{v}_t, \vec{v}_s) &= k_N(\vec{v}, \vec{v}_t) k_N(\vec{v}_t, \vec{v}_s) \\ &\quad + \left(k_w(\vec{v})(\vec{v}, \vec{v}_t) - k_N(\vec{v}, \vec{v}_t) \right) k_w(\vec{v}_t)(\vec{v}_t, \vec{v}_s) \\ &\quad + \left(k_w(\vec{v}_t)(\vec{v}_t, \vec{v}_s) - k_N(\vec{v}_t, \vec{v}_s) \right) k_N(\vec{v}, \vec{v}_t). \end{aligned} \quad (2.132)$$

This means we further split I_4 into

$$I_4 = I_{4,1} + I_{4,2} + I_{4,3}. \quad (2.133)$$

Based on the estimate (2.131), we have

$$I_{4,2} \leq \frac{C}{N} \|g\|_{L^\infty}, \quad (2.134)$$

$$I_{4,3} \leq \frac{C}{N} \|g\|_{L^\infty}. \quad (2.135)$$

Therefore, the only remaining term is $I_{4,1}$. Note we always have $\vec{x} - \epsilon(t_b - s)\vec{v} - \epsilon(s - r)\vec{v}_t \in \Omega$. Hence, we define the change of variable $\vec{y} = (y_1, y_2) = \vec{x} - \epsilon(t_b - s)\vec{v} - \epsilon(s - r)\vec{v}_t$ such that

$$\left| \frac{d\vec{y}}{d\vec{v}_t} \right| = \left\| \begin{pmatrix} \epsilon(s - r) & 0 \\ 0 & \epsilon(s - r) \end{pmatrix} \right\| = \epsilon^2(s - r) \geq \epsilon^2\delta^2. \quad (2.136)$$

Since k_N is bounded and $|\vec{v}_s| \leq 3N$, we estimate

$$\begin{aligned} I_{4,1} &\leq C \left| \int_{|\vec{v}_t| \leq 2N} \int_{|\vec{v}_s| \leq 3N} \int_0^s \mathbf{1}_{\{\vec{x} - \epsilon(t_b - s)\vec{v} - \epsilon(s - r)\vec{v}_t \in \Omega\}} g(\vec{x} - \epsilon(t_b - s)\vec{v} - \epsilon(s - r)\vec{v}_t, \vec{v}_s) dr d\vec{v}_t d\vec{v}_s \right| \\ &\leq \frac{C}{\epsilon\delta} \left| \int_0^s \int_{|\vec{v}_s| \leq 3N} \int_\Omega \mathbf{1}_{\{\vec{y} \in \Omega\}} g(\vec{y}, \vec{v}_s) d\vec{y} d\vec{v}_s dr \right| \\ &\leq \frac{C(1 + N^2)^{\frac{\alpha}{2}} e^{\zeta N^2}}{\epsilon\delta} \left\| \frac{g(\vec{y}, \vec{v}_s)}{w(\vec{v}_s)} \right\|_{L^2} \\ &= \frac{C(1 + N^2)^{\frac{\alpha}{2}} e^{\zeta N^2}}{\epsilon\delta} \left\| \frac{g}{w} \right\|_{L^2}. \end{aligned} \quad (2.137)$$

Summarize all above in Case IV, we obtain

$$I_4 \leq \frac{C}{N} \|g\|_{L^\infty} + \frac{C(1 + N^2)^{\frac{\alpha}{2}} e^{\zeta N^2}}{\epsilon\delta} \left\| \frac{g}{w} \right\|_{L^2}. \quad (2.138)$$

Therefore, we already prove

$$I \leq \left(Ce^{-\frac{\delta}{8}N^2} + \frac{C}{N} + C\delta \right) \|g\|_{L^\infty} + \frac{C(1 + N^2)^{\frac{\alpha}{2}} e^{\zeta N^2}}{\epsilon\delta} \left\| \frac{g}{w} \right\|_{L^2} \quad (2.139)$$

Step 4: Synthesis.

Collecting all above in Step 2 and Step 3, we have shown

$$\|g\|_{L^\infty} \leq \left(Ce^{-\frac{\delta}{8}N^2} + \frac{C}{N} + C\delta \right) \|g\|_{L^\infty} + \frac{C(1 + N^2)^{\frac{\alpha}{2}} e^{\zeta N^2}}{\epsilon\delta} \left\| \frac{g}{w} \right\|_{L^2} + C\|wS\|_{L^\infty} + C|wh|_{L^\infty_-} \quad (2.140)$$

Choosing δ sufficiently small and then taking N sufficiently large such that $Ce^{-\frac{\delta}{8}N^2} + \frac{C}{N} + C\delta \leq \frac{1}{2}$, we have

$$\|g\|_{L^\infty} \leq \frac{C}{\epsilon} \left\| \frac{g}{w} \right\|_{L^2} + C\|wS\|_{L^\infty} + C|wh|_{L^\infty_-}. \quad (2.141)$$

Based on Theorem 2.5, we obtain

$$\begin{aligned} \|g\|_{L^\infty} &\leq C \left(\frac{1}{\epsilon} \left\| \frac{g}{w} \right\|_{L^2} + \|wS\|_{L^\infty} + |wh|_{L^\infty_-} \right) \\ &= C \left(\frac{1}{\epsilon} \|f\|_{L^2} + \|wS\|_{L^\infty} + |wh|_{L^\infty_-} \right) \\ &\leq C \left(\frac{1}{\epsilon^3} \|(\mathbb{I} - \mathbb{P})[S]\|_{L^2} + \frac{1}{\epsilon^2} \|\mathbb{P}[S]\|_{L^2} + \frac{1}{\epsilon^{3/2}} |h|_{L^2_-} + \|wS\|_{L^\infty} + |wh|_{L^\infty_-} \right). \end{aligned} \quad (2.142)$$

This completes the proof. \square

3. ASYMPTOTIC ANALYSIS

In this section, we construct the asymptotic expansion of the equation (1.12).

3.1. Interior Expansion. We define the interior expansion

$$\mathcal{F}^\epsilon \sim \sum_{k=1}^{\infty} \epsilon^k \mathcal{F}_k^\epsilon(\vec{x}, \vec{v}). \quad (3.1)$$

Plugging it into the equation (1.12) and comparing the order of ϵ , we obtain

$$\mathcal{L}[\mathcal{F}_1^\epsilon] = 0, \quad (3.2)$$

$$\mathcal{L}[\mathcal{F}_2^\epsilon] = -\vec{v} \cdot \nabla_x \mathcal{F}_1^\epsilon + \Gamma[\mathcal{F}_1^\epsilon, \mathcal{F}_1^\epsilon], \quad (3.3)$$

$$\mathcal{L}[\mathcal{F}_3^\epsilon] = -\vec{v} \cdot \nabla_x \mathcal{F}_2^\epsilon + \Gamma[\mathcal{F}_1^\epsilon, \mathcal{F}_2^\epsilon] + \Gamma[\mathcal{F}_2^\epsilon, \mathcal{F}_1^\epsilon], \quad (3.4)$$

...

$$\mathcal{L}[\mathcal{F}_k^\epsilon] = -\vec{v} \cdot \nabla_x \mathcal{F}_{k-1}^\epsilon + \sum_{i=1}^{k-1} \Gamma[\mathcal{F}_i^\epsilon, \mathcal{F}_{k-i}^\epsilon]. \quad (3.5)$$

The solvability of

$$\mathcal{L}[\mathcal{F}_k^\epsilon] = -\vec{v} \cdot \nabla_x \mathcal{F}_{k-1}^\epsilon + \sum_{i=1}^{k-1} \Gamma[\mathcal{F}_i^\epsilon, \mathcal{F}_{k-i}^\epsilon] \quad (3.6)$$

requires

$$\int_{\mathbb{R}^2} \left(-\vec{v} \cdot \nabla_x \mathcal{F}_{k-1}^\epsilon + \sum_{i=1}^{k-1} \Gamma[\mathcal{F}_i^\epsilon, \mathcal{F}_{k-i}^\epsilon] \right) \psi(\vec{v}) d\vec{v} = 0 \quad (3.7)$$

for any ψ satisfying $\mathcal{L}[\psi] = 0$. Based on the analysis in [16, 17], each \mathcal{F}_k consists of three parts:

$$\mathcal{F}_k^\epsilon(\vec{x}, \vec{v}) = A_k^\epsilon(\vec{x}, \vec{v}) + B_k^\epsilon(\vec{x}, \vec{v}) + C_k^\epsilon(\vec{x}, \vec{v}), \quad (3.8)$$

where

$$A_k^\epsilon(\vec{x}, \vec{v}) = \sqrt{\mu} \left(A_{k,0}^\epsilon(\vec{x}) + A_{k,1}^\epsilon(\vec{x}) v_1 + A_{k,2}^\epsilon(\vec{x}) v_2 + A_{k,3}^\epsilon(\vec{x}) \left(\frac{|\vec{v}|^2 - 1}{2} \right) \right), \quad (3.9)$$

is the fluid part,

$$B_k^\epsilon(\vec{x}, \vec{v}) = \sqrt{\mu} \left(B_{k,0}^\epsilon(\vec{x}) + B_{k,1}^\epsilon(\vec{x}) v_1 + B_{k,2}^\epsilon(\vec{x}) v_2 + B_{k,3}^\epsilon(\vec{x}) \left(\frac{|\vec{v}|^2 - 1}{2} \right) \right), \quad (3.10)$$

with B_k^ϵ depending on $A_{s,i}^\epsilon$ for $1 \leq s \leq k-1$ and $i = 0, 1, 2, 3$ as

$$B_{k,0}^\epsilon = 0, \quad (3.11)$$

$$B_{k,1}^\epsilon = \sum_{i=1}^{k-1} A_{i,0}^\epsilon A_{k-i,1}^\epsilon, \quad (3.12)$$

$$B_{k,2}^\epsilon = \sum_{i=1}^{k-1} A_{i,0}^\epsilon A_{k-i,2}^\epsilon, \quad (3.13)$$

$$B_{k,3}^\epsilon = \sum_{i=1}^{k-1} \left(A_{i,0}^\epsilon A_{k-i,3}^\epsilon + A_{i,1}^\epsilon A_{k-i,1}^\epsilon + A_{i,2}^\epsilon A_{k-i,2}^\epsilon + \sum_{j=1}^{k-1-i} A_{i,0}^\epsilon (A_{j,1}^\epsilon A_{k-i-j,1}^\epsilon + A_{j,2}^\epsilon A_{k-i-j,2}^\epsilon) \right), \quad (3.14)$$

and $C_k(\vec{x}, \vec{v})$ satisfies

$$\int_{\mathbb{R}^2} \sqrt{\mu} C_k^\epsilon(\vec{x}, \vec{v}) \begin{pmatrix} 1 \\ \vec{v} \\ |\vec{v}|^2 \end{pmatrix} d\vec{v} = 0, \quad (3.15)$$

with

$$\mathcal{L}[C_k^\epsilon] = -\vec{v} \cdot \nabla_x \mathcal{F}_{k-1}^\epsilon + \sum_{i=1}^{k-1} \Gamma[\mathcal{F}_i^\epsilon, \mathcal{F}_{k-i}^\epsilon], \quad (3.16)$$

which can be solved explicitly at any fixed \vec{x} . Hence, we only need to determine the relations satisfied by A_k^ϵ . For convenience, we define

$$A_k^\epsilon = \sqrt{\mu} \left(\rho_k^\epsilon + u_{k,1}^\epsilon v_1 + u_{k,2}^\epsilon v_2 + \theta_k^\epsilon \left(\frac{|\vec{v}|^2 - 1}{2} \right) \right), \quad (3.17)$$

Then the analysis in [16, 17] shows that A_k^ϵ satisfies the equations as follows:

0th order equations:

$$P_1^\epsilon - (\rho_1^\epsilon + \theta_1^\epsilon) = 0, \quad (3.18)$$

$$\nabla_x P_1^\epsilon = 0, \quad (3.19)$$

1st order equations:

$$P_2^\epsilon - (\rho_2^\epsilon + \theta_2^\epsilon + \rho_1^\epsilon \theta_1^\epsilon) = 0, \quad (3.20)$$

$$\vec{u} \cdot \nabla_x \vec{u}_1^\epsilon - \gamma_1 \Delta_x \vec{u}_1^\epsilon + \nabla_x P_2^\epsilon = 0, \quad (3.21)$$

$$\nabla_x \cdot \vec{u}_1^\epsilon = 0, \quad (3.22)$$

$$\vec{u}_1^\epsilon \cdot \nabla_x \theta_1^\epsilon - \gamma_2 \Delta_x \theta_1^\epsilon = 0, \quad (3.23)$$

k^{th} order equations:

$$P_{k+1}^\epsilon - \left(\rho_{k+1}^\epsilon + \theta_{k+1}^\epsilon + \sum_{i=1}^{k+1-i} \rho_i^\epsilon \theta_{k+1-i}^\epsilon \right) = 0, \quad (3.24)$$

$$\sum_{i=1}^k \vec{u}_i^\epsilon \cdot \nabla_x \vec{u}_{k+1-i}^\epsilon - \gamma_1 \Delta_x \vec{u}_k^\epsilon + \nabla_x P_{k+1}^\epsilon = H_{k,1}^\epsilon, \quad (3.25)$$

$$\nabla_x \cdot \vec{u}_k^\epsilon = H_{k,2}^\epsilon, \quad (3.26)$$

$$\sum_{i=1}^k \vec{u}_i^\epsilon \cdot \nabla_x \theta_{k+1-i}^\epsilon - \gamma_2 \Delta_x \theta_k^\epsilon = H_{k,3}^\epsilon, \quad (3.27)$$

where

$$H_{k,j}^\epsilon = G_{k,j}^\epsilon[\vec{x}, \vec{v}; \rho_1^\epsilon, \dots, \rho_{k-1}^\epsilon; \theta_1^\epsilon, \dots, \theta_{k-1}^\epsilon; \vec{u}_1, \dots, \vec{u}_{k-1}^\epsilon], \quad (3.28)$$

is explicit functions depending on lower order terms, and γ_1 and γ_2 are two positive constants. Since in most cases, we are only interested in the leading order terms, so we omit the detailed description of $G_{k,j}^\epsilon$. In order to determine the boundary condition for \vec{u}_k^ϵ , θ_k^ϵ and ρ_k^ϵ , we have to define the boundary layer expansion.

3.2. Boundary Layer Expansion with Geometric Correction. In order to define the boundary layer, we need several substitutions:

Substitution 1:

Define the substitution into polar coordinate $f^\epsilon(x_1, x_2, \vec{v}) \rightarrow f^\epsilon(r, \phi, \vec{v})$ with $(r, \phi, \vec{v}) \in [0, 1) \times [-\pi, \pi) \times \mathbb{R}^2$ as

$$\begin{cases} x_1 &= r \cos \phi, \\ x_2 &= r \sin \phi, \\ \vec{v} &= \vec{v}. \end{cases} \quad (3.29)$$

The equation (1.12) can be rewritten as

$$\begin{cases} \epsilon(\vec{v} \cdot \vec{n}) \frac{\partial f^\epsilon}{\partial r} + \frac{\epsilon}{r}(\vec{v} \cdot \vec{\tau}) \frac{\partial f^\epsilon}{\partial \phi} + \mathcal{L}[f^\epsilon] = \Gamma[f^\epsilon, f^\epsilon], \\ f^\epsilon(1, \phi, \vec{v}) = b^\epsilon(1, \phi, \vec{v}) \quad \text{for } \vec{v} \cdot \vec{n} < 0, \end{cases} \quad (3.30)$$

for \vec{n} the outer normal vector and $\vec{\tau}$ the counterclockwise tangential vector on $\partial\Omega$.

Substitution 2:

We further perform the scaling substitution $f^\epsilon(r, \phi, \vec{v}) \rightarrow f^\epsilon(\eta, \phi, \vec{v})$ with $(\eta, \phi, \vec{v}) \in [0, 1/\epsilon] \times [-\pi, \pi] \times \mathbb{R}^2$ as

$$\begin{cases} \eta &= (1-r)/\epsilon, \\ \phi &= \phi, \\ \vec{v} &= \vec{v}, \end{cases} \quad (3.31)$$

which implies

$$\frac{\partial f^\epsilon}{\partial r} = -\frac{1}{\epsilon} \frac{\partial f^\epsilon}{\partial \eta}, \quad (3.32)$$

Then the equation (1.12) in (η, ϕ, \vec{v}) becomes

$$\begin{cases} -(\vec{v} \cdot \vec{n}) \frac{\partial f^\epsilon}{\partial \eta} + \frac{\epsilon}{1-\epsilon\eta}(\vec{v} \cdot \vec{\tau}) \frac{\partial f^\epsilon}{\partial \phi} + \mathcal{L}[f^\epsilon] = \Gamma[f^\epsilon, f^\epsilon], \\ f^\epsilon(0, \phi, \vec{v}) = b^\epsilon(0, \phi, \vec{v}) \quad \text{for } \vec{v} \cdot \vec{n} < 0. \end{cases} \quad (3.33)$$

Substitution 3:

We define the velocity substitution $f^\epsilon(r, \phi, \vec{v}) \rightarrow f^\epsilon(\eta, \phi, \vec{v}_r)$ with $(\eta, \phi, \vec{v}_r) \in [0, 1/\epsilon] \times [-\pi, \pi] \times \mathbb{R}^2$ as

$$\begin{cases} \eta &= \eta, \\ \phi &= \phi, \\ \vec{v}_r &= -\vec{v}. \end{cases} \quad (3.34)$$

This substitution is for the convenience of Milne problem and specifying the in-flow boundary. Then the equation (1.12) in (η, ϕ, \vec{v}) becomes

$$\begin{cases} (\vec{v}_r \cdot \vec{n}) \frac{\partial f^\epsilon}{\partial \eta} - \frac{\epsilon}{1-\epsilon\eta}(\vec{v}_r \cdot \vec{\tau}) \frac{\partial f^\epsilon}{\partial \phi} + \mathcal{L}[f^\epsilon] = \Gamma[f^\epsilon, f^\epsilon], \\ f^\epsilon(0, \phi, \vec{v}_r) = b^\epsilon(0, \phi, \vec{v}_r) \quad \text{for } \vec{v}_r \cdot \vec{n} > 0. \end{cases} \quad (3.35)$$

Substitution 4:

We further define the velocity decomposition with respect to the normal and tangential directions at boundary as $f^\epsilon(\eta, \phi, v_{r,1}, v_{r,2}) \rightarrow f^\epsilon(\eta, \phi, v_\eta, v_\phi)$ with $(\eta, \phi, v_\eta, v_\phi) \in [0, 1/\epsilon] \times [-\pi, \pi] \times \mathbb{R}^2$

$$\begin{cases} \eta &= \eta, \\ \phi &= \phi, \\ v_\eta &= v_{r,1} \cos \phi + v_{r,2} \sin \phi, \\ v_\phi &= -v_{r,1} \sin \phi + v_{r,2} \cos \phi. \end{cases} \quad (3.36)$$

Denote $\vec{v} = (v_\eta, v_\phi)$. Then the equation (1.12) can be rewritten as

$$\begin{cases} v_\eta \frac{\partial f^\epsilon}{\partial \eta} - \frac{\epsilon}{1-\epsilon\eta} \left(v_\phi \frac{\partial f^\epsilon}{\partial \phi} + v_\phi^2 \frac{\partial f^\epsilon}{\partial v_\eta} - v_\eta v_\phi \frac{\partial f^\epsilon}{\partial v_\phi} \right) + \mathcal{L}[f^\epsilon] = \Gamma[f^\epsilon, f^\epsilon], \\ f^\epsilon(0, \phi, \vec{v}) = b^\epsilon(\phi, \vec{v}) \quad \text{for } v_\eta > 0. \end{cases} \quad (3.37)$$

We define the boundary layer expansion

$$\mathcal{F}^\epsilon \sim \sum_{k=1}^{\infty} \epsilon^k \mathcal{F}_k^\epsilon(\eta, \phi, \vec{v}), \quad (3.38)$$

where \mathcal{F}_k^ϵ can be determined by plugging it into the equation (3.37) and comparing the order of ϵ . In a neighborhood of the boundary, we have

$$v_\eta \frac{\partial \mathcal{F}_1^\epsilon}{\partial \eta} - \frac{\epsilon}{1 - \epsilon\eta} \left(v_\phi^2 \frac{\partial \mathcal{F}_1^\epsilon}{\partial v_\eta} - v_\eta v_\phi \frac{\partial \mathcal{F}_1^\epsilon}{\partial v_\phi} \right) + \mathcal{L}[\mathcal{F}_1^\epsilon] = 0, \quad (3.39)$$

$$v_\eta \frac{\partial \mathcal{F}_2^\epsilon}{\partial \eta} - \frac{\epsilon}{1 - \epsilon\eta} \left(v_\phi^2 \frac{\partial \mathcal{F}_2^\epsilon}{\partial v_\eta} - v_\eta v_\phi \frac{\partial \mathcal{F}_2^\epsilon}{\partial v_\phi} \right) + \mathcal{L}[\mathcal{F}_2^\epsilon] = \frac{1}{1 - \epsilon\eta} v_\phi \frac{\partial \mathcal{F}_1^\epsilon}{\partial \phi} + \Gamma[\mathcal{F}_1^\epsilon, \mathcal{F}_1^\epsilon] + 2\Gamma[\mathcal{F}_1^\epsilon, \mathcal{F}_1^\epsilon], \quad (3.40)$$

...

$$v_\eta \frac{\partial \mathcal{F}_k^\epsilon}{\partial \eta} - \frac{\epsilon}{1 - \epsilon\eta} \left(v_\phi^2 \frac{\partial \mathcal{F}_k^\epsilon}{\partial v_\eta} - v_\eta v_\phi \frac{\partial \mathcal{F}_k^\epsilon}{\partial v_\phi} \right) + \mathcal{L}[\mathcal{F}_k^\epsilon] = \frac{1}{1 - \epsilon\eta} v_\phi \frac{\partial \mathcal{F}_{k-1}^\epsilon}{\partial \phi} + \sum_{i=1}^{k-1} \Gamma[\mathcal{F}_i^\epsilon, \mathcal{F}_{k-i}^\epsilon] + 2 \sum_{i=1}^{k-1} \Gamma[\mathcal{F}_i^\epsilon, \mathcal{F}_{k-i}^\epsilon]. \quad (3.41)$$

3.3. Construction of Asymptotic Expansion. The bridge between interior solution and boundary layer is the boundary condition

$$f^\epsilon(\vec{x}_0, \vec{v}) = b^\epsilon(\vec{x}_0, \vec{v}). \quad (3.42)$$

Plugging the combined expansion

$$f^\epsilon \sim \sum_{k=1}^{\infty} \epsilon^k (\mathcal{F}_k^\epsilon + \mathcal{F}_k^\epsilon), \quad (3.43)$$

into the boundary condition and comparing the order of ϵ , we obtain

$$\mathcal{F}_1^\epsilon + \mathcal{F}_1^\epsilon = b_1, \quad (3.44)$$

$$\mathcal{F}_2^\epsilon + \mathcal{F}_2^\epsilon = b_2, \quad (3.45)$$

...

$$\mathcal{F}_k^\epsilon + \mathcal{F}_k^\epsilon = b_k. \quad (3.46)$$

This is the boundary conditions \mathcal{F}_k^ϵ and \mathcal{F}_k^ϵ need to satisfy. We divide the construction of asymptotic expansion into several steps for each $k \geq 1$:

Step 1: ϵ -Milne Problem.

We solve the ϵ -Milne problem

$$\left\{ \begin{array}{l} v_\eta \frac{\partial g_k^\epsilon}{\partial \eta} + G(\epsilon; \eta) \left(v_\phi^2 \frac{\partial g_k^\epsilon}{\partial v_\eta} - v_\eta v_\phi \frac{\partial g_k^\epsilon}{\partial v_\phi} \right) + \mathcal{L}[g_k^\epsilon] = S_k^\epsilon(\eta, \phi, \vec{v}), \\ g_k^\epsilon(0, \phi, \vec{v}) = h_k^\epsilon(\phi, \vec{v}) \text{ for } v_\eta > 0, \\ \int_{\mathbb{R}^2} v_\eta \sqrt{\mu} g_k^\epsilon(0, \phi, \vec{v}) d\vec{v} = m_f[g_k^\epsilon](\phi), \\ \lim_{\eta \rightarrow \infty} g_k^\epsilon(\eta, \phi, \vec{v}) = g_k^\epsilon(\infty, \phi, \vec{v}), \end{array} \right. \quad (3.47)$$

for $g_k^\epsilon(\eta, \phi, \vec{v})$ with the in-flow boundary data

$$h_k^\epsilon = b_k - (B_k^\epsilon + C_k^\epsilon) \quad (3.48)$$

and source term

$$S_k^\epsilon = \frac{\Upsilon(\epsilon^{1/2}\eta)}{1 - \epsilon\eta} v_\phi \frac{\partial \mathcal{F}_{k-1}^\epsilon}{\partial \phi} + \sum_{i=1}^{k-1} \Gamma[\mathcal{F}_i^\epsilon, \mathcal{F}_{k-i}^\epsilon] + 2 \sum_{i=1}^{k-1} \Gamma[\mathcal{F}_i^\epsilon, \mathcal{F}_{k-i}^\epsilon], \quad (3.49)$$

where

$$G(\epsilon; \eta) = - \frac{\epsilon \Upsilon(\epsilon^{1/2}\eta)}{1 - \epsilon\eta}, \quad (3.50)$$

$$\Upsilon(z) = \begin{cases} 1 & 0 \leq z \leq 1/2, \\ 0 & 3/4 \leq z \leq \infty. \end{cases} \quad (3.51)$$

Here the mass-flux $m_f[g_k^\epsilon](\phi)$ will be determined later. Based on Theorem 4.15, there exist

$$\tilde{h}_k^\epsilon = \sqrt{\mu} \left(\tilde{D}_{k,0}^\epsilon + \tilde{D}_{k,1}^\epsilon v_\eta + \tilde{D}_{k,2}^\epsilon v_\phi + \tilde{D}_{k,3}^\epsilon \left(\frac{|\vec{v}|^2 - 2}{2} \right) \right), \quad (3.52)$$

such that the problem

$$\begin{cases} v_\eta \frac{\partial \mathcal{G}_k^\epsilon}{\partial \eta} + G(\epsilon; \eta) \left(v_\phi^2 \frac{\partial \mathcal{G}_k^\epsilon}{\partial v_\eta} - v_\eta v_\phi \frac{\partial \mathcal{G}_k^\epsilon}{\partial v_\phi} \right) + \mathcal{L}[\mathcal{G}_k^\epsilon] &= S_k^\epsilon(\eta, \phi, \vec{v}), \\ \mathcal{G}_k^\epsilon(0, \phi, \vec{v}) &= h_k^\epsilon(\phi, \vec{v}) - \tilde{h}_k^\epsilon(\phi, \vec{v}) \text{ for } v_\eta > 0, \\ \int_{\mathbb{R}^2} v_\eta \sqrt{\mu} \mathcal{G}_k^\epsilon(0, \phi, \vec{v}) d\vec{v} &= m_f[\mathcal{G}_k^\epsilon](\phi), \\ \lim_{\eta \rightarrow \infty} \mathcal{G}_k^\epsilon(\eta, \phi, \vec{v}) &= 0, \end{cases} \quad (3.53)$$

is well-posed, where we need to specify

$$m_f[\mathcal{G}_k^\epsilon](\phi) = m_f[g_k^\epsilon](\phi) - \int_{\mathbb{R}^2} v_\eta \sqrt{\mu} \tilde{h}_k^\epsilon(\phi, \vec{v}) d\vec{v}. \quad (3.54)$$

Step 2: Definition of Interior Solution and Boundary Layer with Geometric Correction.
Define

$$\mathcal{F}_k^\epsilon = \mathcal{G}_k^\epsilon \cdot \Upsilon_0(\epsilon^{1/2} \eta) \quad (3.55)$$

where \mathcal{G}_k^ϵ the solution of ϵ -Milne problem (3.53) and

$$\Upsilon_0(z) = \begin{cases} 1 & 0 \leq z \leq 1/4, \\ 0 & 1/2 \leq z \leq \infty. \end{cases} \quad (3.56)$$

Naturally, we have

$$\lim_{\eta \rightarrow 0} \mathcal{F}_k^\epsilon(\eta, \phi, \vec{v}) = 0. \quad (3.57)$$

The interior solution

$$\mathcal{F}_k^\epsilon(\vec{x}, \vec{v}) = A_k^\epsilon(\vec{x}, \vec{v}) + B_k^\epsilon(\vec{x}, \vec{v}) + C_k^\epsilon(\vec{x}, \vec{v}), \quad (3.58)$$

where B_k^ϵ and C_k^ϵ are defined in (3.11) and (3.16), and A_k^ϵ satisfies

$$A_k^\epsilon = \sqrt{\mu} \left(\rho_k^\epsilon + u_{k,1}^\epsilon v_1 + u_{k,2}^\epsilon v_2 + \theta_k^\epsilon \left(\frac{|\vec{v}|^2 - 1}{2} \right) \right), \quad (3.59)$$

and

$$P_{k+1}^\epsilon - \left(\rho_{k+1}^\epsilon + \theta_{k+1}^\epsilon + \sum_{i=1}^{k+1-i} \rho_i^\epsilon \theta_{k+1-i}^\epsilon \right) = 0, \quad (3.60)$$

$$\sum_{i=1}^k \vec{u}_i^\epsilon \cdot \nabla_x \vec{u}_{k+1-i}^\epsilon - \gamma_1 \Delta_x \vec{u}_k^\epsilon + \nabla_x P_{k+1}^\epsilon = H_{k,1}^\epsilon, \quad (3.61)$$

$$\nabla_x \cdot \vec{u}_k^\epsilon = H_{k,2}^\epsilon, \quad (3.62)$$

$$\sum_{i=1}^k \vec{u}_i^\epsilon \cdot \nabla_x \theta_{k+1-i}^\epsilon - \gamma_2 \Delta_x \theta_k^\epsilon = H_{k,3}^\epsilon, \quad (3.63)$$

with boundary condition

$$A_{k,0}^\epsilon = \tilde{D}_{k,0}^\epsilon, \quad (3.64)$$

$$A_{k,1}^\epsilon = -\tilde{D}_{k,1}^\epsilon \cos \phi + \tilde{D}_{k,2}^\epsilon \sin \phi, \quad (3.65)$$

$$A_{k,2}^\epsilon = -\tilde{D}_{k,1}^\epsilon \sin \phi - \tilde{D}_{k,2}^\epsilon \cos \phi, \quad (3.66)$$

$$A_{k,3}^\epsilon = \tilde{D}_{k,3}^\epsilon. \quad (3.67)$$

where $\tilde{D}_{k,i}^\epsilon$ comes from the boundary data of ϵ -Milne problem \tilde{h}_k^ϵ . This determines $A_{k,0}^\epsilon$, $A_{k,1}^\epsilon$, $A_{k,2}^\epsilon$ and $A_{k,3}^\epsilon$. Now it is easy to verify the boundary data are satisfied as

$$\mathcal{F}_k^\epsilon + \mathcal{F}_k^\epsilon = b_k. \quad (3.68)$$

Step 3: Boussinesq relation and Vanishing Mass-Flux.

Note that the fluid-type equations satisfied by A_k^ϵ imply Boussinesq relation. In detail,

$$P_{k+1}^\epsilon - \left(\rho_{k+1}^\epsilon + \theta_{k+1}^\epsilon + \sum_{i=1}^{k+1-i} \rho_i^\epsilon \theta_{k+1-i}^\epsilon \right) = 0 \quad (3.69)$$

yields

$$\rho_k^\epsilon + \theta_k^\epsilon + \sum_{i=1}^{k-i} \rho_i^\epsilon \theta_{k-i}^\epsilon = E_k \quad (3.70)$$

which is actually

$$\rho_k^\epsilon + \theta_k^\epsilon = E_k - \sum_{i=1}^{k-i} \rho_i^\epsilon \theta_{k-i}^\epsilon \quad (3.71)$$

for some constant E_k which is free to choose. To enforce this relation, we need to adjust the mass-flux in the ϵ -Milne problem (3.53). Note that the Boussinesq relation (3.71) leads to a given $\tilde{D}_{k,0}^\epsilon(\phi) + \tilde{D}_{k,3}^\epsilon(\phi)$ for any ϕ up to a constant in the ϵ -Milne problem. Theorem 4.8 implies we can always adjust the mass-flux $m_f[\mathcal{G}_k^\epsilon](\phi)$ to guarantee the Boussinesq relation. Based on the proof of Theorem 4.8, we know this can determine the mass-flux $m_f[\mathcal{G}_k^\epsilon](\phi)$ up to a constant.

On the other hand, $f^\epsilon(\vec{x}, \vec{v})$ satisfies the vanishing mass-flux

$$\int_{\partial\Omega} \int_{\mathbb{R}^2} f^\epsilon(\vec{x}, \vec{v}) d\vec{v} d\gamma = 0. \quad (3.72)$$

Then we need

$$\int_{\partial\Omega} \int_{\mathbb{R}^2} (\mathcal{F}_k^\epsilon + \mathcal{F}_k^\epsilon)(\vec{x}, \vec{v}) d\vec{v} d\gamma = 0. \quad (3.73)$$

for any $k \geq 1$. The definition of \mathcal{F}_k^ϵ implies

$$\int_{\partial\Omega} \int_{\mathbb{R}^2} \mathcal{F}_k^\epsilon(\vec{x}, \vec{v}) d\vec{v} d\gamma = 0. \quad (3.74)$$

Then the remaining relation

$$\int_{\partial\Omega} \int_{\mathbb{R}^2} \mathcal{F}_k^\epsilon(\vec{x}, \vec{v}) d\vec{v} d\gamma = 0. \quad (3.75)$$

finally determines the free constant in mass-flux $m_f[\mathcal{G}_k^\epsilon](\phi)$.

In summary, the free mass-flux $m_f[\mathcal{G}_k^\epsilon](\phi)$ can help to enforce two relations: the Boussinesq relation

$$\rho_k^\epsilon + \theta_k^\epsilon = E_k - \sum_{i=1}^{k-i} \rho_i^\epsilon \theta_{k-i}^\epsilon, \quad (3.76)$$

and vanishing mass-flux relation

$$\int_{\partial\Omega} \int_{\mathbb{R}^2} \mathcal{F}_k^\epsilon(\vec{x}, \vec{v}) d\vec{v} d\gamma = 0. \quad (3.77)$$

Therefore, $m_f[\mathcal{G}_k^\epsilon](\phi)$ is completely determined and so are \mathcal{F}_k^ϵ and \mathcal{F}_k^ϵ .

In particular, when $k = 1$, \mathcal{F}_1^ϵ satisfies

$$\left\{ \begin{array}{l} \mathcal{F}_1^\epsilon(\eta, \phi, \vec{v}) = \mathcal{G}_1^\epsilon(\eta, \phi, \vec{v}) \cdot \Upsilon_0(\sqrt{\epsilon\eta}) \\ v_\eta \frac{\partial \mathcal{G}_1^\epsilon}{\partial \eta} + G(\epsilon; \eta) \left(v_\phi^2 \frac{\partial \mathcal{G}_1^\epsilon}{\partial v_\eta} - v_\eta v_\phi \frac{\partial \mathcal{G}_1^\epsilon}{\partial v_\phi} \right) + \mathcal{L}[\mathcal{G}_1^\epsilon] = 0, \\ \mathcal{G}_1^\epsilon(0, \phi, \vec{v}) = b_1(\phi, \vec{v}) - \tilde{h}^\epsilon(\phi, \vec{v}) \text{ for } v_\eta > 0, \\ \int_{\mathbb{R}^2} v_\eta \sqrt{\mu} \mathcal{G}_1^\epsilon(0, \phi, \vec{v}) d\vec{v} = m_f[\mathcal{G}_1^\epsilon], \\ \lim_{\eta \rightarrow \infty} \mathcal{G}_1^\epsilon(\eta, \phi, \vec{v}) = 0, \end{array} \right. \quad (3.78)$$

where

$$\tilde{h}_1^\epsilon = \sqrt{\mu(\vec{v})} \left(\tilde{D}_{1,0}^\epsilon + \tilde{D}_{1,1}^\epsilon v_\eta + \tilde{D}_{1,2}^\epsilon v_\phi + \tilde{D}_{1,3}^\epsilon \left(\frac{|\vec{v}|^2 - 2}{2} \right) \right), \quad (3.79)$$

and \mathcal{F}_1^ϵ satisfies

$$\mathcal{F}_1^\epsilon = \sqrt{\mu} \left(\rho_1^\epsilon + u_{1,1}^\epsilon v_1 + u_{1,2}^\epsilon v_2 + \theta_1^\epsilon \left(\frac{|\vec{v}|^2 - 2}{2} \right) \right), \quad (3.80)$$

with

$$\left\{ \begin{array}{l} \nabla_x(\rho_1^\epsilon + \theta_1^\epsilon) = 0, \\ \vec{u}_1^\epsilon \cdot \nabla_x \vec{u}_1^\epsilon - \gamma_1 \Delta_x \vec{u}_1^\epsilon + \nabla_x P_2^\epsilon = 0, \\ \nabla_x \cdot \vec{u}_1^\epsilon = 0, \\ \vec{u}_1^\epsilon \cdot \nabla_x \theta_1^\epsilon - \gamma_2 \Delta_x \theta_1^\epsilon = 0, \\ \rho_1^\epsilon(\vec{x}_0) = \tilde{D}_{1,0}^\epsilon(\vec{x}_0), \\ u_{1,1}^\epsilon(\vec{x}_0) = -\tilde{D}_{1,1}^\epsilon(\vec{x}_0) \cos \phi + \tilde{D}_{1,2}^\epsilon(\vec{x}_0) \sin \phi, \\ u_{1,2}^\epsilon(\vec{x}_0) = -\tilde{D}_{1,1}^\epsilon(\vec{x}_0) \sin \phi - \tilde{D}_{1,2}^\epsilon(\vec{x}_0) \cos \phi, \\ \theta_1^\epsilon(\vec{x}_0) = \tilde{D}_{1,3}^\epsilon(\vec{x}_0), \end{array} \right. \quad (3.81)$$

where the free mass-flux $m_f[\mathcal{G}_1^\epsilon](\phi)$ is chosen to enforce the Boussinesq relation

$$\rho_1^\epsilon + \theta_1^\epsilon = E_1, \quad (3.82)$$

and vanishing mass-flux relation

$$\int_{\partial\Omega} \int_{\mathbb{R}^2} \mathcal{F}_1^\epsilon(\vec{x}, \vec{v}) d\vec{v} d\gamma = 0. \quad (3.83)$$

Similarly, we can define any \mathcal{F}_k^ϵ and \mathcal{F}_k^ϵ for $k \geq 1$.

4. ϵ -MILNE PROBLEM WITH GEOMETRIC CORRECTION

We consider the ϵ -Milne problem for $g^\epsilon(\eta, \phi, \vec{v})$ in the domain $(\eta, \phi, \vec{v}) \in [0, \infty) \times [-\pi, \pi) \times \mathbb{R}^2$ as

$$\left\{ \begin{array}{l} v_\eta \frac{\partial g^\epsilon}{\partial \eta} + G(\epsilon; \eta) \left(v_\phi^2 \frac{\partial g^\epsilon}{\partial v_\eta} - v_\eta v_\phi \frac{\partial g^\epsilon}{\partial v_\phi} \right) + \mathcal{L}[g^\epsilon] = S^\epsilon(\eta, \phi, \vec{v}), \\ g^\epsilon(0, \phi, \vec{v}) = h^\epsilon(\phi, \vec{v}) \text{ for } v_\eta > 0, \\ \int_{\mathbb{R}^2} v_\eta \sqrt{\mu} g^\epsilon(0, \phi, \vec{v}) d\vec{v} = m_f[g^\epsilon](\phi), \\ \lim_{\eta \rightarrow \infty} g^\epsilon(\eta, \phi, \vec{v}) = g_\infty^\epsilon(\phi, \vec{v}), \end{array} \right. \quad (4.1)$$

where the velocity variables

$$\vec{v} = (v_\eta, v_\phi), \quad (4.2)$$

the standard Maxwellian

$$\mu(\vec{v}) = \frac{1}{2\pi} \exp \left(-\frac{|\vec{v}|^2}{2} \right), \quad (4.3)$$

$m_f[g^\epsilon]$ is the mass-flux given a priori, the limit function

$$g_\infty^\epsilon(\phi, \vec{\mathbf{v}}) = \sqrt{\mu} \left(D_0^\epsilon(\phi) + D_1^\epsilon(\phi)v_\eta + D_2^\epsilon(\phi)v_\phi + D_3^\epsilon(\phi) \frac{|\vec{\mathbf{v}}|^2 - 2}{2} \right), \quad (4.4)$$

the forcing term

$$G(\epsilon; \eta) = - \frac{\epsilon \Upsilon(\epsilon^{1/2} \eta)}{1 - \epsilon \eta}, \quad (4.5)$$

and the cut-off function $\Upsilon(z) \in C^\infty$ is defined as

$$\Upsilon(z) = \begin{cases} 1 & 0 \leq z \leq 1/2, \\ 0 & 3/4 \leq z \leq \infty. \end{cases} \quad (4.6)$$

We assume the boundary data and source term satisfy for $0 \leq \zeta \leq 1/4$ and $\vartheta \geq 0$,

$$\left| \langle \vec{\mathbf{v}} \rangle^\vartheta e^{\zeta |\vec{\mathbf{v}}|^2} h^\epsilon(\phi, \vec{\mathbf{v}}) \right| \leq M, \quad (4.7)$$

and

$$\left| \langle \vec{\mathbf{v}} \rangle^\vartheta e^{\zeta |\vec{\mathbf{v}}|^2} S^\epsilon(\eta, \phi, \vec{\mathbf{v}}) \right| \leq M e^{-K\eta}, \quad (4.8)$$

for $M(\zeta, \vartheta) \geq 0$ and $K(\zeta, \vartheta) > 0$ uniform in ϵ and ϕ , where the Japanese bracket is defined as

$$\langle \vec{\mathbf{v}} \rangle = (1 + |\vec{\mathbf{v}}|^2)^{1/2}. \quad (4.9)$$

We define a potential function $W(\epsilon; \eta)$ as $G(\epsilon; \eta) = -\partial_\eta W(\epsilon; \eta)$ with $W(\epsilon; 0) = 0$. Our main goal is to find

$$\tilde{h}^\epsilon(\phi, \vec{\mathbf{v}}) = \sqrt{\mu} \left(\tilde{D}_0^\epsilon(\phi) + \tilde{D}_1^\epsilon(\phi)v_\eta + \tilde{D}_2^\epsilon(\phi)v_\phi + \tilde{D}_3^\epsilon(\phi) \frac{|\vec{\mathbf{v}}|^2 - 2}{2} \right), \quad (4.10)$$

such that the ϵ -Milne problem for $\mathcal{G}^\epsilon(\eta, \phi, \vec{\mathbf{v}})$ in the domain $(\eta, \phi, \vec{\mathbf{v}}) \in [0, \infty) \times [-\pi, \pi) \times \mathbb{R}^2$ as

$$\left\{ \begin{array}{l} v_\eta \frac{\partial \mathcal{G}^\epsilon}{\partial \eta} + G(\epsilon; \eta) \left(v_\phi^2 \frac{\partial \mathcal{G}^\epsilon}{\partial v_\eta} - v_\eta v_\phi \frac{\partial \mathcal{G}^\epsilon}{\partial v_\phi} \right) + \mathcal{L}[\mathcal{G}^\epsilon] = S^\epsilon(\eta, \phi, \vec{\mathbf{v}}), \\ \mathcal{G}^\epsilon(0, \phi, \vec{\mathbf{v}}) = h^\epsilon(\phi, \vec{\mathbf{v}}) - \tilde{h}^\epsilon(\phi, \vec{\mathbf{v}}) \text{ for } v_\eta > 0, \\ \int_{\mathbb{R}^2} v_\eta \sqrt{\mu} \mathcal{G}^\epsilon(0, \phi, \vec{\mathbf{v}}) d\vec{\mathbf{v}} = m_f[\mathcal{G}^\epsilon](\phi), \\ \lim_{\eta \rightarrow \infty} \mathcal{G}^\epsilon(\eta, \phi, \vec{\mathbf{v}}) = 0, \end{array} \right. \quad (4.11)$$

is well-posed, where

$$m_f[\mathcal{G}^\epsilon](\phi) = m_f[g^\epsilon](\phi) - \int_{\mathbb{R}^2} v_\eta \sqrt{\mu} \tilde{h}^\epsilon(\phi, \vec{\mathbf{v}}) d\vec{\mathbf{v}}. \quad (4.12)$$

For notational simplicity, we omit superscript ϵ and ϕ dependence in g^ϵ and \mathcal{G}^ϵ in this section. The same convention also applies to $G(\epsilon; \eta)$, $W(\epsilon; \eta)$, $S^\epsilon(\eta, \phi, \vec{\mathbf{v}})$ and $h^\epsilon(\phi, \vec{\mathbf{v}})$. It is easy to see the estimates are uniform in ϵ and ϕ . Our analysis is based on the ideas in [1, 7, 19, 18].

In this section, we introduce some special notations to describe the norms in the space $(\eta, \vec{\mathbf{v}}) \in [0, \infty) \times \mathbb{R}^2$. Define the L^2 norm as follows:

$$\|f(\eta)\|_{L^2} = \left(\int_{\mathbb{R}^2} |f(\eta, \vec{\mathbf{v}})|^2 d\vec{\mathbf{v}} \right)^{1/2}, \quad (4.13)$$

$$\|f\|_{L^2 L^2} = \left(\int_0^\infty \int_{\mathbb{R}^2} |f(\eta, \vec{\mathbf{v}})|^2 d\vec{\mathbf{v}} d\eta \right)^{1/2}. \quad (4.14)$$

Define the inner product in $\vec{\mathbf{v}}$ space

$$\langle f, g \rangle(\eta) = \int_{\mathbb{R}^2} f(\eta, \vec{\mathbf{v}}) g(\eta, \vec{\mathbf{v}}) d\vec{\mathbf{v}}. \quad (4.15)$$

Define the weighted L^∞ norm as follows:

$$\|f(\eta)\|_{L^\infty_{\vartheta,\zeta}} = \sup_{\vec{v} \in \mathbb{R}^2} \left(\langle \vec{v} \rangle^\vartheta e^{\zeta |\vec{v}|^2} |f(\eta, \vec{v})| \right), \quad (4.16)$$

$$\|f\|_{L^\infty L^\infty_{\vartheta,\zeta}} = \sup_{(\eta, \vec{v}) \in [0, \infty) \times \mathbb{R}^2} \left(\langle \vec{v} \rangle^\vartheta e^{\zeta |\vec{v}|^2} |f(\eta, \vec{v})| \right), \quad (4.17)$$

$$\|f\|_{L^\infty L^2_\zeta} = \sup_{\eta \in [0, \infty)} \left(\int_{\mathbb{R}^2} |e^{2\zeta |\vec{v}|^2} f(\eta, \vec{v})|^2 d\vec{v} \right)^{1/2}. \quad (4.18)$$

Since the boundary data $h(\vec{v})$ is only defined on $v_\eta > 0$, we naturally extend above definitions on this half-domain as follows:

$$\|h\|_{L^2} = \left(\int_{v_\eta > 0} |h(\vec{v})|^2 d\vec{v} \right)^{1/2}, \quad (4.19)$$

$$\|h\|_{L^\infty_{\vartheta,\zeta}} = \sup_{v_\eta > 0} \left(\langle \vec{v} \rangle^\vartheta e^{\zeta |\vec{v}|^2} |h(\vec{v})| \right). \quad (4.20)$$

Since the null of operator \mathcal{L} is $\mathcal{N} = \sqrt{\mu} \left\{ 1, v_\eta, v_\phi, \frac{|\vec{v}|^2 - 2}{2} \right\} = \{\psi_0, \psi_1, \psi_2, \psi_3\}$, we can decompose the solution as

$$g = w_g + q_g, \quad (4.21)$$

where

$$q_g = \sqrt{\mu} \left(q_{g0} + q_{g1} v_\eta + q_{g2} v_\phi + q_{g3} \frac{|\vec{v}|^2 - 2}{2} \right) = q_{g0} \psi_0 + q_{g1} \psi_1 + q_{g2} \psi_2 + q_{g3} \psi_3 \in \mathcal{N}, \quad (4.22)$$

and

$$w_g \in \mathcal{N}^\perp. \quad (4.23)$$

When there is no confusion, we will simply write $g = w + q$.

Lemma 4.1. *When $\epsilon \leq 1/2$, the force $G(\eta)$ and potential $W(\eta)$ satisfy the following properties:*

1. $W(\eta)$ is an increasing function satisfying for any $\eta > 0$,

$$0 \leq W(\eta) \leq 1, \quad (4.24)$$

2.

$$\lim_{\epsilon \rightarrow 0} W(\infty) = 0. \quad (4.25)$$

3.

$$\int_0^\infty \left(e^{-W(\eta)} - e^{-W(\infty)} \right)^2 d\eta \leq C. \quad (4.26)$$

4.

$$\int_0^\infty G^2(\eta) d\eta \leq C. \quad (4.27)$$

5.

$$\int_0^\infty \int_\eta^\infty G^2(y) dy d\eta \leq C. \quad (4.28)$$

Proof. Since $G(\eta)$ is always zero or negative, we know $W(\eta)$ is increasing. Then we directly compute

$$\begin{aligned} \int_0^\infty |G(\epsilon; \eta)| d\eta &= \int_0^\infty \frac{\epsilon \Upsilon(\epsilon^{1/2} \eta)}{1 - \epsilon \eta} d\eta \leq \int_0^{\frac{3}{4\sqrt{\epsilon}}} \frac{\epsilon}{1 - \epsilon \eta} d\eta \\ &= -\ln(1 - \epsilon \eta) \Big|_0^{\frac{3}{4\sqrt{\epsilon}}} = -\ln \left(1 - \frac{3}{4} \epsilon^{1/2} \right). \end{aligned} \quad (4.29)$$

This naturally implies (4.24) and (4.25). Also, we have

$$\int_0^\infty \left(e^{-W(\eta)} - e^{-W(\infty)} \right)^2 d\eta = e^{-2W(\infty)} \int_0^{\frac{3}{4\sqrt{\epsilon}}} \left(e^{W(\infty)-W(\eta)} - 1 \right)^2 d\eta. \quad (4.30)$$

Then we obtain

$$\int_0^{\frac{3}{4\sqrt{\epsilon}}} e^{W(\infty)-W(\eta)} d\eta = \int_0^{\frac{3}{4\sqrt{\epsilon}}} \exp \left(- \int_\eta^{\frac{3}{4\sqrt{\epsilon}}} G(y) dy \right) d\eta = \int_0^{\frac{3}{4\sqrt{\epsilon}}} \exp \left(\int_\eta^{\frac{3}{4\sqrt{\epsilon}}} \frac{\epsilon \Upsilon(\epsilon^{1/2}y)}{1-\epsilon y} dy \right) d\eta.$$

Note that

$$1 \leq \exp \left(\int_\eta^{\frac{3}{4\sqrt{\epsilon}}} \frac{\epsilon \Upsilon(\epsilon^{1/2}y)}{1-\epsilon y} dy \right) \leq \exp \left(\int_\eta^{\frac{3}{4\sqrt{\epsilon}}} \frac{\epsilon}{1-\epsilon y} dy \right) = \exp \left(-\ln(1-\epsilon y) \Big|_\eta^{\frac{3}{4\sqrt{\epsilon}}} \right) = \frac{1-\epsilon\eta}{1-\frac{3\sqrt{\epsilon}}{4}} \quad (4.31)$$

Hence, we have

$$\left(e^{W(\infty)-W(\eta)} - 1 \right)^2 \leq \max \left\{ \left(\frac{1-\epsilon\eta}{1-\frac{3\sqrt{\epsilon}}{4}} \right)^2 - 1, 2 \left(\frac{1-\epsilon\eta}{1-\frac{3\sqrt{\epsilon}}{4}} - 1 \right) \right\} \leq C\epsilon\eta + C\sqrt{\epsilon}, \quad (4.32)$$

for some constant C independent of ϵ . Then we have

$$\int_0^{\frac{3}{4\sqrt{\epsilon}}} \left(e^{W(\infty)-W(\eta)} - 1 \right)^2 d\eta \leq C \int_0^{\frac{3}{4\sqrt{\epsilon}}} (\epsilon\eta + \sqrt{\epsilon}) d\eta \leq C. \quad (4.33)$$

This proves (4.26). Furthermore,

$$\int_0^\infty G^2(\eta) d\eta = \int_0^\infty \left(\frac{\epsilon \Upsilon(\epsilon^{1/2}\eta)}{1-\epsilon\eta} \right)^2 d\eta \leq \int_0^{\frac{3}{4\sqrt{\epsilon}}} \left(\frac{\epsilon}{1-\epsilon\eta} \right)^2 d\eta = -\epsilon \ln(1-\epsilon\eta) \Big|_0^{\frac{3}{4\sqrt{\epsilon}}} = -\epsilon \ln \left(1 - \frac{3\sqrt{\epsilon}}{4} \right). \quad (4.34)$$

Therefore, (4.27) is obvious. Moreover, we compute

$$\begin{aligned} \int_0^\infty \int_\eta^\infty G^2(y) dy d\eta &= \int_0^\infty \int_\eta^\infty \left(\frac{\epsilon \Upsilon(\epsilon^{1/2}y)}{1-\epsilon y} \right)^2 dy d\eta = \int_0^{\frac{3}{4\sqrt{\epsilon}}} \int_\eta^{\frac{3}{4\sqrt{\epsilon}}} \left(\frac{\epsilon \Upsilon(\epsilon^{1/2}y)}{1-\epsilon y} \right)^2 dy d\eta \\ &\leq \int_0^{\frac{3}{4\sqrt{\epsilon}}} \int_\eta^{\frac{3}{4\sqrt{\epsilon}}} \left(\frac{\epsilon}{1-\epsilon y} \right)^2 dy d\eta = - \int_0^{\frac{3}{4\sqrt{\epsilon}}} \left(\frac{\epsilon}{1-\epsilon y} \Big|_\eta^{\frac{3}{4\sqrt{\epsilon}}} \right) d\eta \\ &= \int_0^{\frac{3}{4\sqrt{\epsilon}}} \left(\frac{\epsilon}{1-\epsilon\eta} - \frac{\epsilon}{1-\frac{3\sqrt{\epsilon}}{4}} \right) d\eta \\ &= -\ln(1-\epsilon\eta) \Big|_0^{\frac{3}{4\sqrt{\epsilon}}} + \frac{\frac{3\sqrt{\epsilon}}{4}}{1-\frac{3\sqrt{\epsilon}}{4}} \leq C. \end{aligned} \quad (4.35)$$

□

Lemma 4.2. *For the operator $\mathcal{L} = \nu - K$, we have the estimates*

$$\nu_0(1 + |\vec{\mathbf{v}}|) \leq \nu(\vec{\mathbf{v}}) \leq \nu_1(1 + |\vec{\mathbf{v}}|), \quad (4.36)$$

$$\langle g, \mathcal{L}[g] \rangle(\eta) = \langle w, \mathcal{L}[w] \rangle(\eta) \geq C \|\sqrt{\nu}w(\eta)\|_{L^2}^2, \quad (4.37)$$

for ν_0 , ν_1 and C positive constants.

Proof. See [10, Chapter 3].

□

4.1. L^2 Estimates.

4.1.1. L^2 Estimates in a finite slab. We first consider the case with zero source term for $g^L(\eta, \vec{v})$ in a finite slab $[0, L] \times \mathbb{R}^2$ as

$$\begin{cases} v_\eta \frac{\partial g^L}{\partial \eta} + G(\eta) \left(v_\phi^2 \frac{\partial g^L}{\partial v_\eta} - v_\eta v_\phi \frac{\partial g^L}{\partial v_\phi} \right) + \mathcal{L}[g^L] = 0, \\ g^L(0, \vec{v}) = h(\vec{v}) \text{ for } v_\eta > 0, \\ g^L(L, R[\vec{v}]) = g^L(L, \vec{v}), \end{cases} \quad (4.38)$$

where

$$R[v_\eta, v_\phi] = (-v_\eta, v_\phi). \quad (4.39)$$

Similarly, we can decompose g^L as

$$g^L = w^L + q^L. \quad (4.40)$$

Lemma 4.3. *There exists a solution of the equation (4.38) satisfying the estimates*

$$\int_0^L \|\sqrt{v} w^L(\eta)\|_{L^2}^2 d\eta \leq C, \quad (4.41)$$

$$\|q^L(\eta)\|_{L^2}^2 \leq C \left(1 + \eta + \|\sqrt{v} w^L(\eta)\|_{L^2}^2 \right), \quad (4.42)$$

where C is a constant independent of L . Also, the solution satisfies the orthogonal relation

$$\langle v_\eta \psi_i, w^L \rangle(\eta) = 0, \text{ for } i = 0, 2, 3. \quad (4.43)$$

Proof. The existence follows from a standard argument by adding penalty term λg^L on the left-hand side of the equation for $0 < \lambda \ll 1$ and estimate along the characteristics. Hence, we concentrate on the estimates. We divide the proof into several steps:

Step 1: Estimate of w^L .

Multiplying g^L on both sides of (4.38) and integrating over $\vec{v} \in \mathbb{R}^2$, we have

$$\frac{1}{2} \frac{d}{d\eta} \langle v_\eta g^L, g^L \rangle + G(\eta) \left\langle v_\phi^2 \frac{\partial g^L}{\partial v_\eta} - v_\eta v_\phi \frac{\partial g^L}{\partial v_\phi}, g^L \right\rangle = -(g^L, \mathcal{L}[g^L]). \quad (4.44)$$

An integration by parts implies

$$\left\langle v_\phi^2 \frac{\partial g^L}{\partial v_\eta} - v_\eta v_\phi \frac{\partial g^L}{\partial v_\phi}, g^L \right\rangle = \frac{1}{2} \left\langle v_\phi^2, \frac{\partial (g^L)^2}{\partial v_\eta} \right\rangle - \frac{1}{2} \left\langle v_\eta v_\phi, \frac{\partial (g^L)^2}{\partial v_\phi} \right\rangle = \frac{1}{2} \langle v_\eta g^L, g^L \rangle. \quad (4.45)$$

Therefore, using Lemma 4.2, we obtain

$$\frac{1}{2} \frac{d}{d\eta} \langle v_\eta g^L, g^L \rangle + \frac{1}{2} G(\eta) \langle v_\eta g^L, g^L \rangle = -(w^L, \mathcal{L}[w^L]). \quad (4.46)$$

Define

$$\alpha(\eta) = \frac{1}{2} \langle v_\eta g^L, g^L \rangle(\eta), \quad (4.47)$$

which implies

$$\frac{d\alpha}{d\eta} + G(\eta)\alpha = -(w^L, \mathcal{L}[w^L]). \quad (4.48)$$

Then we have

$$\alpha(\eta) = \alpha(L) \exp \left(\int_\eta^L G(y) dy \right) + \int_\eta^L \exp \left(- \int_\eta^y G(z) dz \right) \left(\langle w^L, \mathcal{L}[w^L] \rangle(y) \right) dy, \quad (4.49)$$

$$\alpha(\eta) = \alpha(0) \exp \left(- \int_0^\eta G(y) dy \right) + \int_0^\eta \exp \left(\int_y^\eta G(z) dz \right) \left(- \langle w^L, \mathcal{L}[w^L] \rangle(y) \right) dy. \quad (4.50)$$

Since $\alpha(L) = 0$ due to reflexive boundary, (4.49) implies

$$\alpha(\eta) \geq 0. \quad (4.51)$$

By

$$\alpha(0) = \frac{1}{2} \int_{v_\eta > 0} v_\eta (g^L)^2(0) d\vec{v} + \frac{1}{2} \int_{v_\eta < 0} v_\eta (g^L)^2(0) d\vec{v} \leq \frac{1}{2} \int_{v_\eta > 0} v_\eta h^2 d\vec{v} \leq C, \quad (4.52)$$

and (4.50), we obtain

$$\alpha(\eta) \leq C. \quad (4.53)$$

Hence, (4.49) and (4.53) lead to

$$\int_0^L \exp\left(-\int_0^y G(z) dz\right) \left(\langle w^L, \mathcal{L}[w^L] \rangle(y)\right) dy \leq C, \quad (4.54)$$

which, by Lemma 4.1 and Lemma 4.2, further yields

$$\int_0^L \|\sqrt{\nu} w^L(\eta)\|_{L^2}^2 d\eta \leq C \quad (4.55)$$

Step 2: Estimate of q^L .

Multiplying $v_\eta \psi_j$ with $j \neq 1$ on both sides of (4.38) and integrating over $\vec{v} \in \mathbb{R}^2$, we obtain

$$\frac{d}{d\eta} \langle v_\eta^2 \psi_j, g^L \rangle + G(\eta) \left\langle v_\eta \psi_j, v_\phi^2 \frac{\partial g^L}{\partial v_\eta} - v_\eta v_\phi \frac{\partial g^L}{\partial v_\phi} \right\rangle = -\langle v_\eta \psi_j, \mathcal{L}[w^L] \rangle. \quad (4.56)$$

Define $\tilde{q}^L = q^L - q_1^L \psi_1$ and

$$\beta_j(\eta) = \langle v_\eta^2 \psi_j, \tilde{q}^L \rangle(\eta), \quad (4.57)$$

$$\beta(\eta) = \left(\beta_0(\eta), \beta_1(\eta), \beta_2(\eta), \beta_3(\eta) \right)^T \quad (4.58)$$

$$\tilde{\beta}(\eta) = \left(\beta_0(\eta), \beta_2(\eta), \beta_3(\eta) \right)^T. \quad (4.59)$$

By definition, $\beta_1 = 0$. For $j \neq 1$, using integration by parts, we have

$$\frac{d}{d\eta} \langle v_\eta^2 \psi_j, g^L \rangle = G(\eta) \left\langle \frac{\partial}{\partial v_\eta} (v_\eta v_\phi^2 \psi_j) - \frac{\partial}{\partial v_\phi} (v_\eta^2 v_\phi \psi_j), g^L \right\rangle - \langle v_\eta \psi_j, \mathcal{L}[w^L] \rangle, \quad (4.60)$$

which further implies

$$\frac{d\beta_j}{d\eta} = G(\eta) \left\langle \frac{\partial}{\partial v_\eta} (v_\eta v_\phi^2 \psi_j) - \frac{\partial}{\partial v_\phi} (v_\eta^2 v_\phi \psi_j), \tilde{q}^L + q_1^L \psi_1 + w^L \right\rangle - \langle v_\eta \psi_j, \mathcal{L}[w^L] \rangle - \frac{d}{d\eta} \langle v_\eta^2 \psi_j, w^L \rangle. \quad (4.61)$$

Put $\tilde{q}_i^L = q_i^L - \delta_{i1} q_1^L$. Then we can write

$$\left\langle \frac{\partial}{\partial v_\eta} (v_\eta v_\phi^2 \psi_j) - \frac{\partial}{\partial v_\phi} (v_\eta^2 v_\phi \psi_j), \tilde{q}^L \right\rangle(\eta) = \sum_i B_{ji} \tilde{q}_i^L(\eta), \quad (4.62)$$

for $i, j = 0, 2, 3$, where

$$B_{ji} = \left\langle \frac{\partial}{\partial v_\eta} (v_\eta v_\phi^2 \psi_j) - \frac{\partial}{\partial v_\phi} (v_\eta^2 v_\phi \psi_j), \psi_i \right\rangle. \quad (4.63)$$

Moreover,

$$\beta_j(\eta) = \sum_k A_{jk} \tilde{q}_k^L(\eta), \quad (4.64)$$

where

$$A_{jk} = \langle v_\eta^2 \psi_j, \psi_k \rangle, \quad (4.65)$$

is a non-singular matrix such that we can express back

$$\tilde{q}_j^L(\eta) = \sum_k A_{jk}^{-1} \beta_k(\eta). \quad (4.66)$$

Hence, (4.61) can be rewritten as

$$\frac{d\tilde{\beta}}{d\eta} = G(BA^{-1})\tilde{\beta} + D, \quad (4.67)$$

where

$$D_j = G(\eta) \left\langle \frac{\partial}{\partial v_\eta} (v_\eta v_\phi^2 \psi_j) - \frac{\partial}{\partial v_\phi} (v_\eta^2 v_\phi \psi_j), q_1^L \psi_1 + w^L \right\rangle - \langle v_\eta \psi_j, \mathcal{L}[w^L] \rangle - \frac{d}{d\eta} \langle v_\eta^2 \psi_j, w^L \rangle. \quad (4.68)$$

We can solve for β as

$$\tilde{\beta}(\eta) = \exp \left(-W(\eta)BA^{-1} \right) \theta - \zeta(\eta) + \int_0^\eta \exp \left((W(\eta) - W(y))BA^{-1} \right) Z(y) dy, \quad (4.69)$$

where

$$\theta_j = \langle v_\eta^2 \psi_j, g^L \rangle(0), \quad j \neq 1, \quad (4.70)$$

$$\zeta_j(\eta) = \langle v_\eta^2 \psi_j, w^L \rangle(\eta), \quad (4.71)$$

and

$$Z = D + \frac{d\zeta}{d\eta} + G(BA^{-1})\zeta, \quad (4.72)$$

where we use the fact

$$\int_0^\eta \exp \left((W(\eta) - W(y))BA^{-1} \right) \frac{d\zeta}{dy} dy = \zeta(\eta) - \exp \left(-W(\eta)BA^{-1} \right) \zeta(0) - \int_0^\eta G(BA^{-1})\zeta(y) dy. \quad (4.73)$$

Hence, using the boundedness of $W(\eta)$ and BA^{-1} due to Lemma 4.1, we can directly estimate (4.69) to get

$$|\beta_j(\eta)| \leq C|\theta_j| + |\zeta_j(\eta)| + C \int_0^\eta |Z_j(y)| dy \quad \text{for } i = 0, 2, 3. \quad (4.74)$$

By Cauchy's inequality, we obtain

$$|\zeta_j(\eta)| \leq \|\sqrt{\nu}w^L(\eta)\|_{L^2} \quad (4.75)$$

$$|Z_j(\eta)| \leq C \left(\|\sqrt{\nu}w^L(\eta)\|_{L^2} + \|q_1^L(\eta)\psi_1\|_{L^2} \right). \quad (4.76)$$

Multiplying $\sqrt{\mu}$ on both sides of (4.38) and integrating over $\vec{v} \in \mathbb{R}^2$, we have

$$\frac{d}{d\eta} \langle \sqrt{\mu}v_\eta, g^L \rangle = G(\eta) \left\langle \frac{\partial}{\partial v_\eta} (v_\phi^2) - \frac{\partial}{\partial v_\phi} (v_\eta v_\phi), \sqrt{\mu}g^L \right\rangle = -G(\eta) \langle \sqrt{\mu}v_\eta, g^L \rangle, \quad (4.77)$$

which is actually

$$\frac{dq_1^L}{d\eta} = -G(\eta)q_1^L. \quad (4.78)$$

Since $q_1^L(L) = 0$, we have for any $\eta \in [0, L]$,

$$q_1^L(\eta) = 0. \quad (4.79)$$

Also,

$$\langle v_\eta^2 \psi_j, g^L \rangle(0) \leq C \langle |v_\eta| g^L(0), g^L(0) \rangle^{1/2} \langle |v_\eta|^3, \psi_j^2 \rangle^{1/2} \leq C \langle |v_\eta| g^L(0), g^L(0) \rangle^{1/2}, \quad (4.80)$$

$$\langle |v_\eta| g^L(0), g^L(0) \rangle = \int_{v_\eta > 0} \mu v_\eta h^2 - \int_{v_\eta < 0} \mu v_\eta (g^L(0))^2. \quad (4.81)$$

Since

$$\int_{v_\eta > 0} \mu v_\eta h^2 + \int_{v_\eta < 0} \mu v_\eta (g^L(0))^2 = 2\alpha(0) \geq 0, \quad (4.82)$$

we have

$$\theta_j = \langle v_\eta^2 \psi_j, g^L \rangle(0) \leq 2C \int_{v_\eta > 0} \mu v_\eta h^2 \leq C. \quad (4.83)$$

In conclusion, collecting (4.74), (4.75), (4.76), (4.79), and (4.83), we have

$$|\beta_j(\eta)| \leq C \left(1 + \|\sqrt{\nu}w^L(\eta)\|_{L^2} + \int_0^\eta \|\sqrt{\nu}w^L(y)\|_{L^2} dy \right) \quad \text{for } j = 0, 2, 3, \quad (4.84)$$

which further implies

$$|q_j^L(\eta)| \leq C \left(1 + \|\sqrt{\nu}w^L(\eta)\|_{L^2} + \int_0^\eta \|\sqrt{\nu}w^L(y)\|_{L^2} dy \right) \quad \text{for } j = 0, 2, 3, \quad (4.85)$$

and $q_1^L(\eta) = 0$. An application of Cauchy's inequality leads to our desired result.

Step 3: Orthogonal Properties.

In the equation (4.38), multiplying $\sqrt{\mu}$ on both sides and integrating over $\vec{v} \in \mathbb{R}^2$, we have

$$\frac{d}{d\eta} \langle \sqrt{\mu}v_\eta, g^L \rangle = G(\eta) \left\langle \frac{\partial}{\partial v_\eta}(v_\phi^2) - \frac{\partial}{\partial v_\phi}(v_\eta v_\phi), \sqrt{\mu}g^L \right\rangle = -G \langle \sqrt{\mu}v_\eta, g^L \rangle. \quad (4.86)$$

Since $\langle \sqrt{\mu}v_\eta, g^L \rangle(L) = 0$, we have

$$\langle \sqrt{\mu}v_\eta, g^L \rangle(\eta) = 0. \quad (4.87)$$

It is easy to check that

$$\langle v_\eta \psi_i, q^L \rangle = 0 \quad i \neq 1, \quad (4.88)$$

$$\langle v_\eta q^L, q^L \rangle = 0. \quad (4.89)$$

Multiplying ψ_i for $i = 2, 3$ on both sides of (4.38) and integrating over $\vec{v} \in \mathbb{R}^2$, we have

$$\frac{d}{d\eta} \langle v_\eta \psi_i, w^L \rangle = -CG \langle v_\eta \psi_i, w^L \rangle. \quad (4.90)$$

Since $\langle v_\eta \psi_i, w^L(L) \rangle = 0$, then we have

$$\langle v_\eta \psi_i, w^L(\eta) \rangle = 0. \quad (4.91)$$

In summary, we have

$$\langle v_\eta \psi_i, w^L(\eta) \rangle = 0 \quad \text{for } i = 0, 2, 3. \quad (4.92)$$

□

4.1.2. L^2 Estimates in an infinite slab. We consider the case with zero source term and zero mass flux in an infinite slab

$$\left\{ \begin{array}{l} v_\eta \frac{\partial g}{\partial \eta} + G(\eta) \left(v_\phi^2 \frac{\partial g}{\partial v_\eta} - v_\eta v_\phi \frac{\partial g}{\partial v_\phi} \right) + \mathcal{L}[g] = 0, \\ g(0, \vec{v}) = h(\vec{v}) \quad \text{for } v_\eta > 0, \\ \int_{\mathbb{R}^2} v_\eta \sqrt{\mu} g(0, \vec{v}) d\vec{v} = 0 \\ \lim_{\eta \rightarrow \infty} g(\eta, \vec{v}) = g_\infty(\vec{v}). \end{array} \right. \quad (4.93)$$

Lemma 4.4. *There exists a unique solution of the equation (4.93) satisfying the estimate*

$$\|\sqrt{\nu}w\|_{L^2 L^2} \leq C, \quad (4.94)$$

$$|q_{i,\infty}| \leq C, \quad (4.95)$$

$$\|q - q_\infty\|_{L^2 L^2} \leq C, \quad (4.96)$$

where $q_\infty = \sum_{i=0}^3 q_{i,\infty} \psi_i$ and the orthogonal properties:

$$\langle v_\eta \psi_i, w \rangle = 0 \quad \text{for } i = 0, 2, 3. \quad (4.97)$$

Proof. We divide the proof into several steps:

Step 1: Weak convergence.

We can extend the solution g^L by passing $L \rightarrow \infty$. Hence, we can always take weakly convergent subsequence

$$q_i^L(\eta) \rightarrow q_i(\eta) \quad \text{in } L_{\text{loc}}^2([0, \infty)), \quad (4.98)$$

$$w^L \rightarrow w \quad \text{in } L_{\text{loc}}^2([0, \infty), L^2(\mathbb{R}^2)). \quad (4.99)$$

Therefore,

$$g = \sum_{i=0}^3 q_i \psi_i + w, \quad (4.100)$$

is a weak solution of the equation (4.93). Also, by the weak lower semi-continuity, the estimate (4.94) of w is obvious. Also, we can show the orthogonal properties (4.97) when $L \rightarrow \infty$.

Step 2: Estimate of q_∞ .

It is easy to see

$$q_1(\eta) = m_f[g] = 0, \quad (4.101)$$

so we do not need to bother with it. Since $\mathcal{L} : L^2(\mathbb{R}^2) \rightarrow \mathcal{N}^\perp$ with null space \mathcal{N} and image \mathcal{N}^\perp , we have $\tilde{\mathcal{L}} : L^2/\mathcal{N} \rightarrow \mathcal{N}^\perp$ is bijective, where $L^2/\mathcal{N} = \mathcal{N}^\perp$ is the quotient space. Then we can define its inverse, i.e. the pseudo-inverse of \mathcal{L} as $\mathcal{L}^{-1} : \mathcal{N}^\perp \rightarrow \mathcal{N}^\perp$ satisfying $\mathcal{L}\mathcal{L}^{-1}[f] = f$ for any $f \in \mathcal{N}^\perp$.

We intend to multiply $\mathcal{L}^{-1}[v_\eta \psi_i]$ for $i = 2, 3$ on both sides of (4.93) and integrating over \vec{v} . Notice that $v_\eta \psi_2 \in \mathcal{N}^\perp$, but $v_\eta \psi_3 \notin \mathcal{N}^\perp$. Actually, it is easy to verify $v_\eta(\psi_3 - \psi_0) \in \mathcal{N}^\perp$. To avoid introducing new notation, we still use ψ_3 to denote $\psi_3 - \psi_0$ in the following proof and it is easy to see there is no confusion. Then we get

$$\frac{d}{d\eta} \langle \mathcal{L}^{-1}[\psi_i v_\eta], v_\eta g \rangle + G(\eta) \left\langle \mathcal{L}^{-1}[\psi_i v_\eta], \left(v_\phi^2 \frac{\partial g}{\partial v_\eta} - v_\eta v_\phi \frac{\partial g}{\partial v_\phi} \right) \right\rangle = - \langle \mathcal{L}^{-1}[\psi_i v_\eta], \mathcal{L}[w] \rangle. \quad (4.102)$$

Since \mathcal{L} is self-adjoint, combining with the orthogonal properties, we have

$$\langle \mathcal{L}^{-1}[\psi_i v_\eta], \mathcal{L}[w] \rangle = \left\langle \mathcal{L} \left[\mathcal{L}^{-1}[\psi_i v_\eta] \right], w \right\rangle = \langle \psi_i v_\eta, w \rangle = 0. \quad (4.103)$$

Therefore, we have

$$\frac{d}{d\eta} \langle v_\eta \mathcal{L}^{-1}[\psi_i v_\eta], g \rangle + G(\eta) \left\langle \mathcal{L}^{-1}[\psi_i v_\eta], \left(v_\phi^2 \frac{\partial g}{\partial v_\eta} - v_\eta v_\phi \frac{\partial g}{\partial v_\phi} \right) \right\rangle = 0. \quad (4.104)$$

Since $\psi_1 \in \mathcal{N}$ and $\mathcal{L}^{-1}[\psi_i v_\eta] \in \mathcal{N}^\perp$, we have

$$\langle \psi_1, \mathcal{L}^{-1}[\psi_i v_\eta] \rangle = 0. \quad (4.105)$$

For $i, k = 2, 3$, put

$$N_{i,k} = \langle v_\eta \mathcal{L}^{-1}[\psi_i v_\eta], \psi_k \rangle, \quad (4.106)$$

$$P_{i,k} = \left\langle \left(v_\phi^2 \frac{\partial}{\partial v_\eta} - v_\eta v_\phi \frac{\partial}{\partial v_\phi} + v_\eta \right) \mathcal{L}^{-1}[\psi_i v_\eta], \psi_k \right\rangle. \quad (4.107)$$

Thus,

$$\Omega_i = \langle v_\eta \mathcal{L}^{-1}[\psi_i v_\eta], q \rangle = \sum_{k=2}^3 N_{i,k} q_k(\eta), \quad (4.108)$$

and

$$\left\langle \left(v_\phi^2 \frac{\partial}{\partial v_\eta} - v_\eta v_\phi \frac{\partial}{\partial v_\phi} + v_\eta \right) \mathcal{L}^{-1}[\psi_i v_\eta], q \right\rangle = \sum_{k=2}^3 P_{i,k} q_k(\eta). \quad (4.109)$$

Since matrix N is invertible (see [12]), from (4.104) and integration by parts, we have for $i = 2, 3$,

$$\begin{aligned} \frac{d\Omega_i}{d\eta} = & -\frac{d}{d\eta} \langle v_\eta \mathcal{L}^{-1}[\psi_i v_\eta], w \rangle \\ & + \sum_{k=2}^3 G(\eta)(PN^{-1})_{ik} \Omega_k + G(\eta) \left\langle \left(v_\phi^2 \frac{\partial}{\partial v_\eta} - v_\eta v_\phi \frac{\partial}{\partial v_\phi} + v_\eta \right) \mathcal{L}^{-1}[\hat{\psi}_i v_\eta], w \right\rangle. \end{aligned} \quad (4.110)$$

Denote

$$\Omega' = \exp \left(W(\eta) P N^{-1} \right) \Omega. \quad (4.111)$$

Let $\hat{\psi} = (\psi_2, \psi_3)^T$, we can solve

$$\begin{aligned} \Omega'(\eta) = & \langle v_\eta \mathcal{L}^{-1}[\hat{\psi} v_\eta], g \rangle(0) - \exp \left(W(\eta) P N^{-1} \right) \langle v_\eta \mathcal{L}^{-1}[\hat{\psi} v_\eta], w \rangle(\eta) \\ & + \int_0^\eta \exp \left(W(y) P N^{-1} \right) G(y) \left(\left\langle \left(v_\phi^2 \frac{\partial}{\partial v_\eta} - v_\eta v_\phi \frac{\partial}{\partial v_\phi} + v_\eta \right) \mathcal{L}^{-1}[\hat{\psi} v_\eta], w \right\rangle(y) \right. \\ & \left. + \sum_{k=2}^3 P N^{-1} \langle v_\eta \mathcal{L}^{-1}[\hat{\psi} v_\eta], w \rangle(y) \right) dy. \end{aligned} \quad (4.112)$$

By a similar method as in the proof of Lemma 4.3 to bound $\theta_i(0)$, we can show

$$\langle v_\eta \mathcal{L}^{-1}[\hat{\psi} v_\eta], g \rangle(0) < \infty. \quad (4.113)$$

Since $w \in L^2([0, \infty) \times \mathbb{R}^2)$, considering $W(\eta)$ and $P N^{-1}$ are bounded, and $G(\eta) \in L^\infty$, we define

$$\begin{aligned} \Omega'_\infty = & \langle v_\eta \mathcal{L}^{-1}[\hat{\psi} v_\eta], g \rangle(0) + \int_0^\infty \exp \left(W(y) P N^{-1} \right) G(y) \left(\left\langle \left(v_\phi^2 \frac{\partial}{\partial v_\eta} - v_\eta v_\phi \frac{\partial}{\partial v_\phi} + v_\eta \right) \mathcal{L}^{-1}[\hat{\psi} v_\eta], w \right\rangle(y) \right. \\ & \left. + \sum_{k=2}^3 P N^{-1} \langle v_\eta \mathcal{L}^{-1}[\hat{\psi} v_\eta], w \rangle(y) \right) dy. \end{aligned} \quad (4.114)$$

Let $\hat{q} = (q_2, q_3)^T$. Then we can define

$$\hat{q}_\infty = N^{-1} \exp \left(-W(\infty) P N^{-1} \right) \Omega'_\infty. \quad (4.115)$$

Finally, we consider q_0 . Multiplying ψ_1 on both sides of (4.93) and integrating over \vec{v} , we obtain

$$\frac{d}{d\eta} \langle \psi_1 g, v_\eta \rangle = -G(\eta) \left\langle \psi_1, \left(v_\phi^2 \frac{\partial g}{\partial v_\eta} - v_\eta v_\phi \frac{\partial g}{\partial v_\phi} \right) \right\rangle = G(\eta) \langle g, (v_\phi^2 - v_\eta^2) \sqrt{\mu} \rangle. \quad (4.116)$$

Then integrating over $[0, \eta]$, we obtain

$$\begin{aligned} \langle \psi_1 g, v_\eta \rangle(\eta) &= \langle \psi_1 g, v_\eta \rangle(0) + \int_0^\eta G(y) \langle g, (v_\phi^2 - v_\eta^2) \sqrt{\mu} \rangle dy \\ &= \langle \psi_1 g, v_\eta \rangle(0) + \int_0^\eta G(y) \langle w, (v_\phi^2 - v_\eta^2) \sqrt{\mu} \rangle(y) dy. \end{aligned} \quad (4.117)$$

Since $w \in L^2([0, \infty) \times \mathbb{R}^2)$ and we can also bound $\langle v_\eta g, v_\eta \rangle(0)$, we have

$$\lim_{\eta \rightarrow \infty} \langle \psi_1 g, v_\eta \rangle(\eta) = \langle \psi_1 g, v_\eta \rangle(0) + \int_0^\infty G(y) \langle w, (v_\phi^2 - v_\eta^2) \sqrt{\mu} \rangle(y) dy \quad (4.118)$$

exists. Note that

$$\lim_{\eta \rightarrow \infty} \langle \psi_1 q_{1,\infty}, v_\eta \rangle(\eta) = \lim_{\eta \rightarrow \infty} \langle \psi_1 q_{2,\infty}, v_\eta \rangle(\eta) = 0. \quad (4.119)$$

Then we define

$$q_{0,\infty} = \frac{\lim_{\eta \rightarrow \infty} \langle \psi_1 g, v_\eta \rangle (\eta) - q_{3,\infty} \langle \psi_1 \psi_3, v_\eta \rangle}{\langle v_\eta \psi_0, v_\eta \rangle}. \quad (4.120)$$

Then to summarize all above, we have defined q_∞ which satisfies $|q_{i,\infty}| \leq C$ for $i = 0, 1, 2, 3$.

Step 3: L^2 Decay of w .

The orthogonal property and zero mass-flux imply

$$\langle v_\eta q, w \rangle (\eta) = \sum_{k=0}^3 \langle v_\eta \psi_k, w \rangle (\eta) = 0. \quad (4.121)$$

The oddness and zero mass-flux imply

$$\langle v_\eta q, q \rangle (\eta) = 0. \quad (4.122)$$

Therefore, we deduce that

$$\langle v_\eta g, g \rangle (\eta) = \langle v_\eta w, w \rangle (\eta). \quad (4.123)$$

Multiplying $e^{2K_0\eta}g$ on both sides of (4.93) and integrating over \vec{v} , we obtain

$$\frac{1}{2} \frac{d}{d\eta} \left(e^{2K_0\eta+W(\eta)} \langle v_\eta w, w \rangle \right) - e^{2K_0\eta+W(\eta)} \left(K_0 \langle v_\eta w, w \rangle - \langle w, \mathcal{L}w \rangle \right) = 0. \quad (4.124)$$

Since

$$\langle \mathcal{L}[w], w \rangle \geq \nu_0 \langle (1 + |\vec{v}|)w, w \rangle, \quad (4.125)$$

and $W(\eta)$ is bounded, for K_0 sufficiently small, we have

$$\langle \mathcal{L}[w], w \rangle - \langle K_0 v_\eta w, w \rangle \geq C \langle w, w \rangle. \quad (4.126)$$

Then by a similar argument as in Lemma 4.3, we can show

$$\int_0^\infty e^{2K_0\eta} \langle \nu w, w \rangle (\eta) d\eta \leq C. \quad (4.127)$$

Step 4: Estimate of $q - q_\infty$.

We first consider $\hat{q} = (q_2, q_3)^T$, which satisfies

$$\hat{q}(\eta) = N^{-1} \exp \left(-W(\eta)PN^{-1} \right) \Omega'(\eta), \quad (4.128)$$

Let

$$\delta = \left\langle v_\eta \mathcal{L}^{-1}[\hat{\psi}v_\eta], g \right\rangle (0) \quad (4.129)$$

$$\Delta = G \left(\left\langle \left(v_\phi^2 \frac{\partial}{\partial v_\eta} - v_\eta v_\phi \frac{\partial}{\partial v_\phi} + v_\eta \right) \mathcal{L}^{-1}[\hat{\psi}v_\eta], w \right\rangle + \sum_{k=2}^3 PN^{-1} \left\langle v_\eta \mathcal{L}^{-1}[\hat{\psi}v_\eta], w \right\rangle \right) \quad (4.130)$$

Then we have

$$\begin{aligned} \hat{q}(\eta) &= N^{-1} \exp \left(-W(\eta)PN^{-1} \right) \delta - N^{-1} \left\langle v_\eta \mathcal{L}^{-1}[\hat{\psi}v_\eta], w \right\rangle (\eta) \\ &\quad + \int_0^\eta \exp \left((W(y) - W(\eta))PN^{-1} \right) \Delta(y) dy. \end{aligned} \quad (4.131)$$

Also, we know

$$\hat{q}_\infty = N^{-1} \exp \left(-W(\infty)PN^{-1} \right) \delta + \int_0^\infty \exp \left((W(y) - W(\infty))PN^{-1} \right) \Delta(y) dy. \quad (4.132)$$

Then we have

$$\begin{aligned} \hat{q}(\eta) - \hat{q}_\infty &= N^{-1} \left(\exp \left(-W(\eta)PN^{-1} \right) - \exp \left(-W(\infty)PN^{-1} \right) \right) \delta - N^{-1} \left\langle v_\eta \mathcal{L}^{-1}[\hat{\psi}v_\eta], w \right\rangle (\eta) \\ &\quad + N^{-1} \left(\exp \left(-W(\eta)PN^{-1} \right) - \exp \left(-W(\infty)PN^{-1} \right) \right) \int_0^\infty \exp \left(W(y)PN^{-1} \right) \Delta(y) dy \\ &\quad + N^{-1} \int_\eta^\infty \exp \left((W(y) - W(\eta))PN^{-1} \right) \Delta(y) dy. \end{aligned} \quad (4.133)$$

Then we have

$$\begin{aligned} &\| \hat{q} - \hat{q}_\infty \|_{L^2 L^2} \\ &\leq \left\| \left\| N^{-1} \left(\exp \left(-W(\eta)PN^{-1} \right) - \exp \left(-W(\infty)PN^{-1} \right) \right) \delta \right\|_{L^2 L^2} + \left\| N^{-1} \left\langle v_\eta \mathcal{L}^{-1}[\hat{\psi}v_\eta], w \right\rangle \right\|_{L^2 L^2} \right. \\ &\quad + \left\| \left\| N^{-1} \left(\exp \left(-W(\eta)PN^{-1} \right) - \exp \left(-W(\infty)PN^{-1} \right) \right) \int_0^\infty \exp \left(W(y)PN^{-1} \right) \Delta(y) dy \right\|_{L^2 L^2} \right. \\ &\quad \left. + \left\| \left\| N^{-1} \int_\eta^\infty \exp \left((W(y) - W(\eta))PN^{-1} \right) \Delta(y) dy \right\|_{L^2 L^2} \right\|. \end{aligned} \quad (4.134)$$

We need to estimate each term on the right-hand side of (4.134). By Lemma 4.1, we have

$$\left\| \left\| N^{-1} \left(\exp \left(-W(\eta)PN^{-1} \right) - \exp \left(-W(\infty)PN^{-1} \right) \right) \delta \right\|_{L^2 L^2}^2 \right. \quad (4.135)$$

$$\leq C \delta \left\| e^{-W(\eta)} - e^{-W(\infty)} \right\|_{L^2 L^2} \leq C. \quad (4.136)$$

Since $w \in L^2([0, \infty) \times \mathbb{R}^2)$, we have

$$\left\| \left\| N^{-1} \left\langle v_\eta \mathcal{L}^{-1}[\hat{\psi}v_\eta], w \right\rangle \right\|_{L^2 L^2} \leq C \|w\|_{L^2 L^2} \leq C. \quad (4.137)$$

Similarly, we can show

$$\begin{aligned} &\left\| \left\| N^{-1} \left(\exp \left(-W(\eta)PN^{-1} \right) - \exp \left(-W(\infty)PN^{-1} \right) \right) \int_0^\infty \exp \left(W(y)PN^{-1} \right) \Delta(y) dy \right\|_{L^2 L^2} \right. \\ &\leq C \left\| e^{-W(\eta)} - e^{-W(\infty)} \right\|_{L^2 L^2} \|r\|_{L^2 L^2} \leq C. \end{aligned} \quad (4.138)$$

For the last term, we have to resort to exponential decay of w . We estimate

$$\begin{aligned} &\left\| \left\| N^{-1} \int_\eta^\infty \exp \left((W(y) - W(\eta))PN^{-1} \right) \Delta(y) dy \right\|_{L^2 L^2} \right. \\ &\leq C \int_0^\infty \left(\int_\eta^\infty \Delta(y) dy \right)^2 d\eta \leq \int_0^\infty \left(\int_\eta^\infty e^{-2K_0 y} dy \right) \left(\int_\eta^\infty w^2(y) e^{2K_0 y} dy \right) d\eta \\ &\leq \int_0^\infty C e^{-2K_0 \eta} d\eta \leq C. \end{aligned} \quad (4.139)$$

Collecting all above, we have

$$\| \hat{q} - \hat{q}_\infty \|_{L^2 L^2} \leq C. \quad (4.140)$$

Then we turn to q_0 . We have

$$q_0(\eta) = \frac{\langle \psi_1 g, v_\eta \rangle (\eta) - q_3(\eta) \langle \psi_1 \psi_3, v_\eta \rangle}{\langle v_\eta \psi_0, v_\eta \rangle}, \quad (4.141)$$

where

$$\langle \psi_1 g, v_\eta \rangle (\eta) = \langle \psi_1 g, v_\eta \rangle (0) + \int_0^\eta G(y) \langle w, (v_\phi^2 - v_\eta^2) \sqrt{\mu} \rangle (y) dy. \quad (4.142)$$

Also, we have

$$q_{0,\infty}(\eta) = \frac{\lim_{\eta \rightarrow \infty} \langle \psi_1 g, v_\eta \rangle(\eta) - q_{3,\infty} \langle \psi_1 \psi_3, v_\eta \rangle}{\langle v_\eta \psi_0, v_\eta \rangle}, \quad (4.143)$$

where

$$\lim_{\eta \rightarrow \infty} \langle \psi_1 g, v_\eta \rangle(\eta) = \langle \psi_1 g, v_\eta \rangle(0) + \int_0^\infty G(y) \langle w, (v_\phi^2 - v_\eta^2) \sqrt{\mu} \rangle(y) dy. \quad (4.144)$$

Therefore, we have

$$q_0(\eta) - q_{0,\infty} = \frac{\int_\eta^\infty G(y) \langle w, (v_\phi^2 - v_\eta^2) \sqrt{\mu} \rangle(y) dy - (q_3(\eta) - q_{3,\infty}) \langle \psi_1 \psi_3, v_\eta \rangle}{\langle v_\eta \psi_0, v_\eta \rangle} \quad (4.145)$$

Then we can naturally estimate

$$\|q_0 - q_{0,\infty}\|_{L^2 L^2} \leq C \left\| \int_\eta^\infty G(y) \langle w, (v_\phi^2 - v_\eta^2) \sqrt{\mu} \rangle(y) dy \right\|_{L^2 L^2} + C \|q_3(\eta) - q_{3,\infty}\|_{L^2 L^2} \quad (4.146)$$

$\|q_3(\eta) - q_{3,\infty}\|_{L^2 L^2}$ is bounded due to the estimate of $\|\hat{q}(\eta) - \hat{q}_\infty\|_{L^2 L^2}$. Then by Cauchy's inequality and Lemma 4.1, we obtain

$$\left\| \int_\eta^\infty G(y) \langle w, (v_\phi^2 - v_\eta^2) \sqrt{\mu} \rangle(y) dy \right\|_{L^2 L^2} \leq \int_0^\infty \left(\int_\eta^\infty G(y) \|r(y)\|_{L^2} dy \right)^2 d\eta \quad (4.147)$$

$$\leq \|r\|_{L^2 L^2} \int_0^\infty \int_\eta^\infty G^2(y) dy d\eta \leq C. \quad (4.148)$$

Therefore, we have shown

$$\|q_0(\eta) - q_{0,\infty}\|_{L^2 L^2} \leq C. \quad (4.149)$$

In summary, we prove that

$$\|q - q_\infty\|_{L^2 L^2} \leq C. \quad (4.150)$$

Step 5: Uniqueness.

If g_1 and g_2 are two solutions of (4.93), define $g' = g_1 - g_2$. Then g' satisfies the equation

$$\begin{cases} v_\eta \frac{\partial g'}{\partial \eta} + G(\eta) \left(v_\phi^2 \frac{\partial g'}{\partial v_\eta} - v_\eta v_\phi \frac{\partial g'}{\partial v_\phi} \right) + \mathcal{L}[g'] &= 0, \\ g'(0, \vec{v}) &= 0 \text{ for } v_\eta > 0, \\ \int_{\mathbb{R}^2} v_\eta \sqrt{\mu} g'(0, \vec{v}) d\vec{v} &= 0, \\ \lim_{\eta \rightarrow \infty} g'(\eta, \vec{v}) &= g'_\infty(\vec{v}), \end{cases} \quad (4.151)$$

Similarly, we can define $g' = w' + q'$. Define the linearized entropy as

$$H[g'](\eta) = \langle v_\eta g', g' \rangle(\eta). \quad (4.152)$$

Multiplying g' on both sides of (4.151) and integrating over \vec{v} , we get

$$\frac{1}{2} \frac{d}{d\eta} \langle v_\eta g', g' \rangle = \langle w', \mathcal{L}[w'] \rangle - \frac{1}{2} G(\eta) \langle v_\eta g', g' \rangle. \quad (4.153)$$

Hence, we have

$$\frac{1}{2} \frac{d}{d\eta} \left(e^W \langle v_\eta g', g' \rangle \right) = -e^W \langle w', \mathcal{L}[w'] \rangle, \quad (4.154)$$

which implies $e^W H[g']$ is decreasing. Furthermore, we have

$$e^{W(\eta)} H[g'](\eta) = H[g'](0) - \int_0^\eta e^{W(y)} \langle w', \mathcal{L}[w'] \rangle(y) dy < \infty. \quad (4.155)$$

Hence, we can take a subsequence such that $\|\sqrt{\nu}w'(\eta_n)\|_{L^2}$ goes to zero. Then we can always assume $q'(\eta_n)$ goes to q'_∞ . Therefore, we have

$$e^{W(\eta_n)}H[g'](\eta_n) \rightarrow \langle v_\eta q'_\infty, q'_\infty \rangle. \quad (4.156)$$

Since $m_f[g'] = 0$, we naturally obtain

$$e^{W(\eta_n)}H[g'](\eta_n) \rightarrow 0 \quad \text{as } \eta_n \rightarrow \infty. \quad (4.157)$$

Hence, we have

$$e^{W(\eta)}H[g'](\eta) \geq 0, \quad (4.158)$$

and

$$e^{W(\eta)}H[g'](\eta) \rightarrow 0 \quad \text{as } \eta \rightarrow \infty. \quad (4.159)$$

In (4.154), integrating over $[0, \infty)$, we achieve

$$-\int_{v_\eta < 0} v_\eta (g')^2(0) d\vec{v} + \int_0^\infty e^{W(\eta)} \langle w', \mathcal{L}[w'] \rangle (\eta) d\eta = \int_{v_\eta > 0} v_\eta (g')^2(0) d\vec{v} = 0. \quad (4.160)$$

Hence, we have

$$\int_{v_\eta < 0} v_\eta (g')^2(0) d\vec{v} = \int_0^\infty \langle w', \mathcal{L}[w'] \rangle (\eta) d\eta = 0, \quad (4.161)$$

which implies $g'(0) = 0$ and $w' = 0$. Hence, $g' = q'$ and satisfies

$$v_\eta \frac{\partial g'}{\partial \eta} + G(\eta) \left(v_\phi^2 \frac{\partial g'}{\partial v_\eta} - v_\eta v_\phi \frac{\partial g'}{\partial v_\phi} \right) = 0. \quad (4.162)$$

$m_f[g'] = 0$ implies $q'_1 = 0$. Therefore, multiplying $v_\eta \psi_i$ for $i \neq 1$ on both sides of (4.162) and integrating over \vec{v} , we obtain a linear system on q'_k for $k = 1, 2, 3$ with initial data zero, which possesses a unique solution zero. This means $g' = 0$. Hence, the solution is unique. \square

4.1.3. L^2 Estimates with general source term and non-vanishing mass-flux. We consider the Milne problem with general source term and non-vanishing mass-flux.

$$\begin{cases} v_\eta \frac{\partial g}{\partial \eta} + G(\eta) \left(v_\phi^2 \frac{\partial g}{\partial v_\eta} - v_\eta v_\phi \frac{\partial g}{\partial v_\phi} \right) + \mathcal{L}[g] = S, \\ g(0, \vec{v}) = h(\vec{v}) \quad \text{for } v_\eta > 0, \\ \int_{\mathbb{R}^2} v_\eta \sqrt{\mu} g(0, \vec{v}) d\vec{v} = m_f[g] \\ \lim_{\eta \rightarrow \infty} g(\eta, \vec{v}) = g_\infty(\vec{v}). \end{cases} \quad (4.163)$$

Lemma 4.5. *There exists a unique solution of the equation (4.163) satisfying the estimate*

$$\|\sqrt{\nu}w\|_{L^2 L^2} \leq C, \quad (4.164)$$

$$|q_{i,\infty}| \leq C, \quad (4.165)$$

$$\|q - q_\infty\|_{L^2 L^2} \leq C, \quad (4.166)$$

where $q_\infty = \sum_{i=0}^3 q_{i,\infty} \psi_i$.

Proof. For the non-vanishing mass flux problem, we can see $\hat{g} = g - m_f[g] \sqrt{\mu} e^{-\eta} v_\eta$ satisfies the ϵ -Milne problem with zero mass flux with the source term

$$\hat{S} = S + m_f[g] \sqrt{\mu} \left(v_\eta^2 - G(\eta) v_\phi^2 \right) e^{-\eta}, \quad (4.167)$$

and the boundary data

$$\hat{h} = h - m_f[g] \sqrt{\mu} v_\eta. \quad (4.168)$$

Therefore, we only need to consider the case with general source term and zero mass-flux. However, if $\int_{\mathbb{R}^2} \sqrt{\mu} S(\eta, \vec{v}) d\vec{v} \neq 0$, the mass-flux is not conserved when η changes. The construction of solutions can be divided into several steps:

Step 1: Decomposition of the source term.

We decompose the source term as

$$S = S_Q + S_W, \quad (4.169)$$

where $S_Q \in \mathcal{N}$ is the kernel part and $S_W = S - S_Q \in \mathcal{N}^\perp$.

Step 2: Construction of g_1 .

We first solve the problem with source term S_W as

$$\left\{ \begin{array}{l} v_\eta \frac{\partial g_1}{\partial \eta} + G(\eta) \left(v_\phi^2 \frac{\partial g_1}{\partial v_\eta} - v_\eta v_\phi \frac{\partial g_1}{\partial v_\phi} \right) + \mathcal{L}[g_1] = S_W, \\ g_1(0, \vec{v}) = h(\vec{v}) \text{ for } v_\eta > 0, \\ \int_{\mathbb{R}^2} v_\eta \sqrt{\mu} g_1(0, \vec{v}) d\vec{v} = 0 \\ \lim_{\eta \rightarrow \infty} g_1(\eta, \vec{v}) = g_{1,\infty}(\vec{v}). \end{array} \right. \quad (4.170)$$

In this case, we apply similar techniques as in the analysis of $S = 0$ case. All the results can be generalized in a natural way. Hence, we know g_1 is well-posed.

Step 3: Construction of g_2 .

We try to find a function g_2 such that

$$\mathcal{L} \left[v_\eta \frac{\partial g_2}{\partial \eta} + G(\eta) \left(v_\phi^2 \frac{\partial g_2}{\partial v_\eta} - v_\eta v_\phi \frac{\partial g_2}{\partial v_\phi} \right) + S_Q \right] = 0. \quad (4.171)$$

which further means

$$\int_{\mathbb{R}^2} \psi_i \left(v_\eta \frac{\partial g_2}{\partial \eta} + G(\eta) \left(v_\phi^2 \frac{\partial g_2}{\partial v_\eta} - v_\eta v_\phi \frac{\partial g_2}{\partial v_\phi} \right) + S_Q \right) d\vec{v} = 0. \quad (4.172)$$

for $i = 0, 1, 2, 3$. Consider

$$S_Q = \sqrt{\mu} \left(a(\eta) + \vec{b}(\eta) \cdot \vec{v} + c(\eta) |\vec{v}|^2 \right). \quad (4.173)$$

We make an ansatz that

$$g_2 = \sqrt{\mu} \left(A(\eta) v_\eta + B_1(\eta) + B_2(\eta) v_\eta v_\phi + C(\eta) v_\eta |\vec{v}|^2 \right). \quad (4.174)$$

Plugging this ansatz into the equation (4.172), we obtain a system of linear ordinary differential equations which is well-posed. Hence, we can naturally obtain g_2 . Furthermore, g_2 decays exponentially with respect to η as long as the boundary data are taken properly.

Step 4: Construction of g_3 .

We may directly verify

$$\mathcal{L} \left[v_\eta \frac{\partial g_2}{\partial \eta} + G(\eta) \left(v_\phi^2 \frac{\partial g_2}{\partial v_\eta} - v_\eta v_\phi \frac{\partial g_2}{\partial v_\phi} \right) + \mathcal{L}[g_2] + S_Q \right] = 0 \quad (4.175)$$

Then we may define g_3 as the solution of the equation

$$\left\{ \begin{array}{l} v_\eta \frac{\partial g_3}{\partial \eta} + G(\eta) \left(v_\phi^2 \frac{\partial g_3}{\partial v_\eta} - v_\eta v_\phi \frac{\partial g_3}{\partial v_\phi} \right) + \mathcal{L}[g_3] = v_\eta \frac{\partial g_2}{\partial \eta} + G(\eta) \left(v_\phi^2 \frac{\partial g_2}{\partial v_\eta} - v_\eta v_\phi \frac{\partial g_2}{\partial v_\phi} \right) + \mathcal{L}[g_2] + S_Q, \\ g_3(0, \vec{v}) = -g_2(0, \vec{v}) \text{ for } v_\eta > 0, \\ \int_{\mathbb{R}^2} v_\eta \sqrt{\mu} g_3(0, \vec{v}) d\vec{v} = 0 \\ \lim_{\eta \rightarrow \infty} g_3(\eta, \vec{v}) = g_{3,\infty}(\vec{v}). \end{array} \right. \quad (4.176)$$

We can obtain g_3 is well-posed.

Step 5: Construction of g_4 .

We may directly verify $g_4 = g_2 + g_3$ satisfies the equation

$$\left\{ \begin{array}{l} v_\eta \frac{\partial g_4}{\partial \eta} + G(\eta) \left(v_\phi^2 \frac{\partial g_4}{\partial v_\eta} - v_\eta v_\phi \frac{\partial g_4}{\partial v_\phi} \right) + \mathcal{L}[g_4] = S_Q, \\ g_4(0, \vec{\mathbf{v}}) = h(\vec{\mathbf{v}}) \text{ for } v_\eta > 0, \\ \int_{\mathbb{R}^2} v_\eta \sqrt{\mu} g_4(0, \vec{\mathbf{v}}) d\vec{\mathbf{v}} = 0 \\ \lim_{\eta \rightarrow \infty} g_4(\eta, \vec{\mathbf{v}}) = g_{4,\infty}(\vec{\mathbf{v}}). \end{array} \right. \quad (4.177)$$

In summary, we know $g = g_1 + g_4$ satisfies the equation (4.163) with zero mass-flux and is well-posed. \square

Lemma 4.6. *Assume (4.7) and (4.8) hold. There exists a unique solution $g(\eta, \vec{\mathbf{v}})$ to the ϵ -Milne problem (4.1) satisfying*

$$\|g - g_\infty\|_{L^2 L^2} \leq C. \quad (4.178)$$

Proof. Taking $g_\infty = q_\infty$, we can naturally obtain the desired result. \square

Then we turn to the construction of \tilde{h} and the well-posedness of the equation (4.11).

Theorem 4.7. *Assume (4.7) and (4.8) hold. There exists \tilde{h} satisfying the condition (4.10) such that there exists a unique solution $\mathcal{G}(\eta, \vec{\mathbf{v}})$ to the ϵ -Milne problem (4.11) satisfying*

$$\|\mathcal{G}\|_{L^2 L^2} \leq C. \quad (4.179)$$

Proof. The key part is the construction of \tilde{h} . Our main idea is to find $\tilde{h} \in \mathcal{N}$ such that the equation

$$\left\{ \begin{array}{l} v_\eta \frac{\partial \tilde{g}}{\partial \eta} + G(\eta) \left(v_\phi^2 \frac{\partial \tilde{g}}{\partial v_\eta} - v_\eta v_\phi \frac{\partial \tilde{g}}{\partial v_\phi} \right) + \mathcal{L}[\tilde{g}] = 0, \\ \tilde{g}(0, \vec{\mathbf{v}}) = \tilde{h}(\vec{\mathbf{v}}) \text{ for } v_\eta > 0, \\ \int_{\mathbb{R}^2} v_\eta \sqrt{\mu} \tilde{g}(0, \vec{\mathbf{v}}) d\vec{\mathbf{v}} = \int_{\mathbb{R}^2} v_\eta \sqrt{\mu} \tilde{h}(\vec{\mathbf{v}}) d\vec{\mathbf{v}}, \\ \lim_{\eta \rightarrow \infty} \tilde{g}(\eta, \vec{\mathbf{v}}) = \tilde{g}_\infty(\vec{\mathbf{v}}), \end{array} \right. \quad (4.180)$$

for $\tilde{g}(\eta, \vec{\mathbf{v}})$ is well-posed, where

$$\tilde{g}_\infty(\vec{\mathbf{v}}) = g_\infty(\vec{\mathbf{v}}) = q_{0,\infty} \psi_0 + q_{1,\infty} \psi_1 + q_{2,\infty} \psi_2 + q_{3,\infty} \psi_3, \quad (4.181)$$

is given by the equation of g . Note that

$$\tilde{h}^\epsilon(\vec{\mathbf{v}}) = \tilde{D}_0^\epsilon \psi_0 + \tilde{D}_1^\epsilon \psi_1 + \tilde{D}_2^\epsilon \psi_2 + \tilde{D}_3^\epsilon \psi_3. \quad (4.182)$$

We consider the endomorphism T in \mathcal{N} defined as $T : \tilde{h} \rightarrow T[\tilde{h}] = \tilde{g}_\infty$. Therefore, we only need to study the matrix of T at the basis $\{\psi_0, \psi_1, \psi_2, \psi_3\}$. It is easy to check when $\tilde{h} = \psi_0$ and $\tilde{h} = \psi_3$, T is an identity mapping, i.e.

$$T[\psi_0] = \psi_0 \quad (4.183)$$

$$T[\psi_3] = \psi_3 \quad (4.184)$$

Multiplying $\sqrt{\mu}$ on both sides of (4.180) and integrating over $\vec{\mathbf{v}} \in \mathbb{R}^2$ imply conserved mass-flux, which further leads to

$$T[\psi_1] = \psi_1 \quad (4.185)$$

The main obstacle is when $\tilde{h} = \psi_2$. In this case, define $\tilde{g}' = \tilde{g} - \psi_2$. Then \tilde{g}' satisfies the equation

$$\left\{ \begin{array}{l} v_\eta \frac{\partial \tilde{g}'}{\partial \eta} + G(\eta) \left(v_\phi^2 \frac{\partial \tilde{g}'}{\partial v_\eta} - v_\eta v_\phi \frac{\partial \tilde{g}'}{\partial v_\phi} \right) + \mathcal{L}[\tilde{g}'] = G(\eta) \sqrt{\mu} v_\eta v_\phi, \\ \tilde{g}'(0, \vec{\mathbf{v}}) = 0 \text{ for } v_\eta > 0, \\ \int_{\mathbb{R}^2} v_\eta \sqrt{\mu} \tilde{g}'(0, \vec{\mathbf{v}}) d\vec{\mathbf{v}} = 0, \\ \lim_{\eta \rightarrow \infty} \tilde{g}'(\eta, \vec{\mathbf{v}}) = \tilde{g}'_\infty(\vec{\mathbf{v}}). \end{array} \right. \quad (4.186)$$

Although $G(\eta)\sqrt{\mu}v_\eta v_\phi$ does not decay exponentially, based on Lemma 4.1, L^1 and L^2 norm of G can be sufficiently small as $\epsilon \rightarrow 0$ and $G(\eta)\sqrt{\mu}v_\eta v_\phi \in \mathcal{N}$. Using a natural extension of Lemma 4.4 for $\mathcal{L}[S] = 0$, we know $|\tilde{q}'_\infty|$ is also sufficiently small, where \tilde{q}' is the projection of \tilde{g}' on \mathcal{N} . Note that we do not need exponential decay of source term in order to show the bound of \tilde{q}_∞ . This means

$$T[\psi_0, \psi_1, \psi_2, \psi_3] = [\psi_0, \psi_1, \psi_2, \psi_3] \begin{pmatrix} 1 & 0 & \tilde{q}'_{0,\infty} & 0 \\ 0 & 1 & \tilde{q}'_{1,\infty} & 0 \\ 0 & 0 & 1 + \tilde{q}'_{2,\infty} & 0 \\ 0 & 0 & \tilde{q}'_{3,\infty} & 1 \end{pmatrix} \quad (4.187)$$

For ϵ sufficiently small, this matrix is invertible, which means T is bijective. Therefore, we can always find \tilde{h} such that $\tilde{g}_\infty = g_\infty$, which is desired. Then by Lemma 4.6 and superposition property, when define $\mathcal{G}^\epsilon = g^\epsilon - \tilde{g}$, the theorem naturally follows. \square

In the ϵ -Milne problem (4.11), even if the boundary data and source term are determined, we still have the freedom to choose mass-flux $m_f[g]$. Next theorem shows that we can adjust the mass-flux $m_f[\mathcal{G}]$ to obtain desired properties of \tilde{h} .

Theorem 4.8. *Assume (4.7) and (4.8) hold. In the ϵ -Milne problem (4.11), for any constant C_0 , there exists an $m_f[g]$ such that $\tilde{\gamma} = \tilde{D}_0^\epsilon + \tilde{D}_3^\epsilon = C_0$.*

Proof. The key point is to study the equation for \bar{g} as

$$\left\{ \begin{array}{l} v_\eta \frac{\partial \bar{g}}{\partial \eta} + G(\epsilon; \eta) \left(v_\phi^2 \frac{\partial \bar{g}}{\partial v_\eta} - v_\eta v_\phi \frac{\partial \bar{g}}{\partial v_\phi} \right) + \mathcal{L}[\bar{g}] = 0, \\ \bar{g}(0, \vec{v}) = 0 \text{ for } v_\eta > 0, \\ \int_{\mathbb{R}^2} v_\eta \sqrt{\mu} \bar{g}(0, \vec{v}) d\vec{v} = m_f[\bar{g}], \\ \lim_{\eta \rightarrow \infty} \bar{g}(\eta, \vec{v}) = \bar{g}_\infty(\vec{v}), \end{array} \right. \quad (4.188)$$

where

$$\bar{g}_\infty(\vec{v}) = E_0\psi_0 + E_1\psi_1 + E_2\psi_2 + E_3\psi_3, \quad (4.189)$$

with zero boundary data and source term but non-vanishing mass-flux, i.e. $m_f[\bar{g}] \neq 0$. We claim $E_1 + E_3 \neq 0$. If this claim is true, then by superposition property, in the equation (4.1), we can obtain the desired $\gamma = g_{0,\infty} + g_{3,\infty}$ by adding a multiple of the equation (4.188). Then as in the proof of Theorem 4.7, we know the endomorphism T leads to $g_{0,\infty} + g_{3,\infty} = \tilde{D}_0^\epsilon + \tilde{D}_3^\epsilon$. Then our work is done.

Next, we prove this claim by contradiction. Let us assume the claim is not true, i.e. $E_1 + E_3 = 0$ for some $m_f[\bar{g}] \neq 0$. We decompose $\bar{g} = \bar{w} + \bar{q}$. By the construction in the proof of Lemma 4.5, we know

$$0 < \|\bar{w}\|_{L^2 L^2} \leq C. \quad (4.190)$$

Note there the first inequality is valid since we can directly verify $\bar{g} \in \mathcal{N}$ cannot be a solution. Define the linearized entropy as

$$H[\bar{g}](\eta) = \langle v_\eta \bar{g}, \bar{g} \rangle(\eta). \quad (4.191)$$

Multiplying \bar{g} on both sides of (4.188) and integrating over \vec{v} , we get

$$\frac{1}{2} \frac{d}{d\eta} \langle v_\eta \bar{g}, \bar{g} \rangle = \langle \bar{w}, \mathcal{L}[\bar{w}] \rangle - \frac{1}{2} G(\eta) \langle v_\eta \bar{g}, \bar{g} \rangle. \quad (4.192)$$

Hence, we have

$$\frac{1}{2} \frac{d}{d\eta} \left(e^W \langle v_\eta \bar{g}, \bar{g} \rangle \right) = -e^W \langle \bar{w}, \mathcal{L}[\bar{w}] \rangle, \quad (4.193)$$

which implies $e^W H[\bar{g}]$ is decreasing. Furthermore, we have

$$e^{W(\eta)} H[\bar{g}](\eta) = H[\bar{g}](0) - \int_0^\eta e^{W(y)} \langle \bar{w}, \mathcal{L}[\bar{w}] \rangle(y) dy < \infty. \quad (4.194)$$

Hence, we can take a subsequence such that $\|\sqrt{\nu}\bar{w}(\eta_n)\|_{L^2}$ goes to zero. Then we can always assume $\bar{q}(\eta_n)$ goes to \bar{q}_∞ . Therefore, we have

$$e^{W(\eta_n)}H[\bar{g}](\eta_n) \rightarrow \langle v_\eta \bar{q}_\infty, \bar{q}_\infty \rangle = 2E_1(E_0 + E_3) = 0. \quad (4.195)$$

Then we naturally obtain

$$e^{W(\eta_n)}H[\bar{g}](\eta_n) \rightarrow 0 \quad \text{as } \eta_n \rightarrow \infty. \quad (4.196)$$

Hence, we have

$$e^{W(\eta)}H[\bar{g}](\eta) \geq 0, \quad (4.197)$$

and

$$e^{W(\eta)}H[\bar{g}](\eta) \rightarrow 0 \quad \text{as } \eta \rightarrow \infty. \quad (4.198)$$

In (4.193), integrating over $[0, \infty)$, we achieve

$$-\int_{v_\eta < 0} v_\eta \bar{g}^2(0) d\vec{v} + \int_0^\infty e^{W(\eta)} \langle \bar{w}, \mathcal{L}[\bar{w}] \rangle(\eta) d\eta = \int_{v_\eta > 0} v_\eta \bar{g}^2(0) d\vec{v} = 0. \quad (4.199)$$

Hence, we have

$$\int_{v_\eta < 0} v_\eta \bar{g}^2(0) d\vec{v} = \int_0^\infty \langle \bar{w}, \mathcal{L}[\bar{w}] \rangle(\eta) d\eta = 0, \quad (4.200)$$

which implies $\bar{g}(0) = 0$ and $\bar{w} = 0$. This contradicts (4.190). Therefore, the claim is valid. \square

4.2. L^∞ Estimates.

4.2.1. *Mild formulation in a finite slab.* Consider the ϵ -transport problem for $g^L(\eta, \vec{v})$ in a finite slab

$$\begin{cases} v_\eta \frac{\partial g^L}{\partial \eta} + G(\eta) \left(v_\phi^2 \frac{\partial g^L}{\partial v_\eta} - v_\eta v_\phi \frac{\partial g^L}{\partial v_\phi} \right) + \nu g^L &= Q(\eta, \vec{v}), \\ g^L(0, \vec{v}) &= h(\vec{v}) \quad \text{for } v_\eta > 0, \\ g^L(L, R[\vec{v}]) &= g^L(\vec{v}), \end{cases} \quad (4.201)$$

We define the characteristics starting from $(\eta(0), v_\eta(0), v_\phi(0))$ as $(\eta(s), v_\eta(s), v_\phi(s))$ defined by

$$\frac{d\eta}{ds} = v_\eta \quad (4.202)$$

$$\frac{dv_\eta}{ds} = G(\eta) v_\phi^2 \quad (4.203)$$

$$\frac{dv_\phi}{ds} = -G(\eta) v_\eta v_\phi \quad (4.204)$$

which leads to

$$v_\eta^2(s) + v_\phi^2(s) = C_1, \quad (4.205)$$

$$v_\phi(s) e^{-W(s)} = C_2, \quad (4.206)$$

where C_1 and C_2 are two constants depending on the starting point. Along the characteristics, the equation (4.201) can be rewritten as

$$v_\eta \frac{\partial g}{\partial \eta} + \nu g = Q. \quad (4.207)$$

Define the energy

$$E(\eta, \vec{v}) = v_\eta^2(\eta) + v_\phi^2(\eta). \quad (4.208)$$

and

$$v'_\phi(\eta, \vec{v}; \eta') = v_\phi e^{W(\eta') - W(\eta)}. \quad (4.209)$$

For $E \geq v_\phi'^2$, define

$$v_\eta'(\eta, \vec{v}; \eta') = \sqrt{E - v_\phi'^2(\eta, \vec{v}; \eta')}, \quad (4.210)$$

$$\vec{v}'(\eta, \eta') = (v_\eta'(\eta, \vec{v}; \eta'), v_\phi'(\eta, \vec{v}; \eta')), \quad (4.211)$$

$$R[\vec{v}'(\eta, \eta')] = (-v_\eta'(\eta, \vec{v}; \eta'), v_\phi'(\eta, \vec{v}; \eta')). \quad (4.212)$$

Basically, this means (η, v_η, v_ϕ) and $(\eta', v_\eta', v_\phi')$ are on the same characteristics. Moreover, define an implicit function $\eta^+(\eta, \vec{v})$ by the equation

$$E(\eta, \vec{v}) = v_\phi'^2(\eta, \vec{v}; \eta^+). \quad (4.213)$$

We know (η^+, \vec{v}) at the axis $v_\eta = 0$ is on the same characteristics as (η, \vec{v}) . Finally put

$$G_{\eta, \eta'} = \int_{\eta'}^{\eta} \frac{\nu(\vec{v}'(\eta, y))}{v_\eta'(\eta, \vec{v}, y)} dy, \quad (4.214)$$

$$R[G_{\eta, \eta'}] = \int_{\eta'}^{\eta} \frac{\nu(R[\vec{v}'(\eta, y)])}{v_\eta'(\eta, \vec{v}, y)} dy. \quad (4.215)$$

We can rewrite the solution to the equation (4.201) along the characteristics as follows:

Case I:

For $v_\eta > 0$,

$$g^L(\eta, \vec{v}) = h(\vec{v}'(\eta, \vec{v}; 0)) \exp(-G_{\eta, 0}) + \int_0^{\eta} \frac{Q(\eta', \vec{v}(\eta, \vec{v}; \eta'))}{v_\eta'(\eta, \vec{v}, \eta')} \exp(-G_{\eta, \eta'}) d\eta'. \quad (4.216)$$

Case II:

For $v_\eta < 0$ and $|E(\eta, \vec{v})| \geq v_\phi'(\eta, \vec{v}; L)$,

$$\begin{aligned} g^L(\eta, \vec{v}) &= h(\vec{v}'(\eta, \vec{v}; 0)) \exp(-G_{L, 0} - R[G_{L, \eta}]) \\ &+ \left(\int_0^L \frac{Q(\eta', \vec{v}(\eta, \vec{v}; \eta'))}{v_\eta'(\eta, \vec{v}, \eta')} \exp(-G_{L, \eta'} - R[G_{L, \eta}]) d\eta' \right. \\ &\quad \left. + \int_\eta^L \frac{Q(\eta', R[\vec{v}'(\eta, \vec{v}; \eta')])}{v_\eta'(\eta, \vec{v}, \eta')} \exp(R[G_{\eta, \eta'}]) d\eta' \right). \end{aligned} \quad (4.217)$$

Case III:

For $v_\eta < 0$ and $|E(\eta, \vec{v})| \leq v_\phi'(\eta, \vec{v}; L)$,

$$\begin{aligned} g^L(\eta, \vec{v}) &= h(\vec{v}'(\eta, \vec{v}; 0)) \exp(-G_{\eta^+, 0} - R[G_{\eta^+, \eta}]) \\ &+ \left(\int_0^{\eta^+} \frac{Q(\eta', \vec{v}(\eta, \vec{v}; \eta'))}{v_\eta'(\eta, \vec{v}, \eta')} \exp(-G_{\eta^+, \eta'} - R[G_{\eta^+, \eta}]) d\eta' \right. \\ &\quad \left. + \int_\eta^{\eta^+} \frac{Q(\eta', R[\vec{v}'(\eta, \vec{v}; \eta')])}{v_\eta'(\eta, \vec{v}, \eta')} \exp(R[G_{\eta, \eta'}]) d\eta' \right). \end{aligned} \quad (4.218)$$

4.2.2. *Mild formulation in an infinite slab.* Consider the ϵ -transport problem for $g(\eta, \vec{v})$ in an infinite slab

$$\begin{cases} v_\eta \frac{\partial g}{\partial \eta} + G(\eta) \left(v_\phi^2 \frac{\partial g}{\partial v_\eta} - v_\eta v_\phi \frac{\partial g}{\partial v_\phi} \right) + \nu g &= Q(\eta, \vec{v}), \\ g(0, \vec{v}) &= h(\vec{v}) \text{ for } v_\eta > 0, \\ \lim_{\eta \rightarrow \infty} g(\eta, \vec{v}) &= g_\infty(\vec{v}), \end{cases} \quad (4.219)$$

We can define the solution via taking limit $L \rightarrow \infty$ in (4.216), (4.217) and (4.218) as follows:

$$g(\eta, \vec{v}) = \mathcal{A}[h(\vec{v})] + \mathcal{T}[Q(\eta, \vec{v})], \quad (4.220)$$

where

Case I:

For $v_\eta > 0$,

$$\mathcal{A}[h(\vec{v})] = h(\vec{v}'(\eta, \vec{v}; 0)) \exp(-G_{\eta,0}), \quad (4.221)$$

$$\mathcal{T}[Q(\eta, \vec{v})] = \int_0^\eta \frac{Q(\eta', \vec{v}'(\eta, \vec{v}; \eta'))}{v'_\eta(\eta, \vec{v}, \eta')} \exp(-G_{\eta,\eta'}) d\eta'. \quad (4.222)$$

Case II:

For $v_\eta < 0$ and $|E(\eta, \vec{v})| \geq v'_\phi(\eta, \vec{v}; \infty)$,

$$\mathcal{A}[h(\vec{v})] = 0, \quad (4.223)$$

$$\mathcal{T}[Q(\eta, \vec{v})] = \int_\eta^\infty \frac{Q(\eta', R[\vec{v}'(\eta, \vec{v}; \eta')])}{v'_\eta(\eta, \vec{v}, \eta')} \exp(R[G_{\eta,\eta'}]) d\eta'. \quad (4.224)$$

Case III:

For $v_\eta < 0$ and $|E(\eta, \vec{v})| \leq v'_\phi(\eta, \vec{v}; \infty)$,

$$\mathcal{A}[h(\vec{v})] = h(\vec{v}'(\eta, \vec{v}; 0)) \exp(-G_{\eta^+,0} - R[G_{\eta^+,\eta}]), \quad (4.225)$$

$$\begin{aligned} \mathcal{T}[Q(\eta, \vec{v})] = & \left(\int_0^{\eta^+} \frac{Q(\eta', \vec{v}'(\eta, \vec{v}; \eta'))}{v'_\eta(\eta, \vec{v}, \eta')} \exp(-G_{\eta^+,\eta'} - R[G_{\eta^+,\eta}]) d\eta' \right. \\ & \left. + \int_\eta^{\eta^+} \frac{Q(\eta', R[\vec{v}'(\eta, \vec{v}; \eta')])}{v'_\eta(\eta, \vec{v}, \eta')} \exp(R[G_{\eta,\eta'}]) d\eta' \right). \end{aligned} \quad (4.226)$$

Notice that

$$\lim_{L \rightarrow \infty} \exp(-G_{L,\eta}) = 0, \quad (4.227)$$

for $v_\eta < 0$ and $|E(\eta, \vec{v})| \leq v'_\phi(\eta, \vec{v}; \infty)$. Hence, above derivation is valid. In order to achieve the estimate of g , we need to control $\mathcal{A}[h]$ and $\mathcal{T}[Q]$.

4.2.3. Preliminaries.

Lemma 4.9. *There is a positive $0 < \beta < \nu_0$ such that for any $\vartheta \geq 0$ and $0 \leq \zeta \leq 1/4$,*

$$\|e^{\beta\eta} \mathcal{A}[h]\|_{L^\infty_{\vartheta,\zeta}} \leq C \|h\|_{L^\infty_{\vartheta,\zeta}}. \quad (4.228)$$

Proof. Based on Lemma 4.2, we know

$$\frac{\nu(\vec{v}'(\eta, y))}{v'_\eta(\eta, \vec{v}, y)} \geq \nu_0 \quad (4.229)$$

$$\frac{\nu(R[\vec{v}'(\eta, y)])}{v'_\eta(\eta, \vec{v}, y)} \geq \nu_0 \quad (4.230)$$

It follows that

$$\exp(-G_{\eta,0}) \leq e^{-\beta\eta} \quad (4.231)$$

$$\exp(-G_{\eta^+,0} - R[G_{\eta^+,\eta}]) \leq e^{-\beta\eta} \quad (4.232)$$

Then our results are obvious. \square

Lemma 4.10. *For any integer $\vartheta \geq 0$, $0 \leq \zeta \leq 1/4$ and $\beta \leq \nu_0/2$, there is a constant C such that*

$$\|\mathcal{T}[Q]\|_{L^\infty L^\infty_{\vartheta,\zeta}} \leq C \left\| \frac{Q}{\nu} \right\|_{L^\infty L^\infty_{\vartheta,\zeta}}. \quad (4.233)$$

Moreover, we have

$$\|e^{\beta\eta} \mathcal{T}[Q]\|_{L^\infty L^\infty_{\vartheta,\zeta}} \leq C \left\| \frac{e^{\beta\eta} Q}{\nu} \right\|_{L^\infty L^\infty_{\vartheta,\zeta}}. \quad (4.234)$$

Proof. The first inequality is a special case of the second one, so we only need to prove the second inequality. For $v_\eta > 0$ case, we have

$$\beta(\eta - \eta') - G_{\eta, \eta'} \leq \beta(\eta - \eta') - \frac{\nu_0(\eta - \eta')}{2} - \frac{G_{\eta, \eta'}}{2} \leq -\frac{G_{\eta, \eta'}}{2}. \quad (4.235)$$

It is natural that

$$\int_0^\eta \frac{\nu(\vec{v}'(\eta, \eta'))}{v'_\eta(\eta, \vec{v}, \eta')} \exp(\beta(\eta - \eta') - G_{\eta, \eta'}) d\eta' \leq \int_0^\infty \exp\left(-\frac{z}{2}\right) dz = 2. \quad (4.236)$$

Then we estimate

$$\begin{aligned} \left| \langle \vec{v} \rangle^\vartheta e^{\zeta|\vec{v}|^2} e^{\beta\eta} \mathcal{T}[Q] \right| &\leq e^{\beta\eta} \int_0^\eta \langle \vec{v} \rangle^\vartheta e^{\zeta|\vec{v}|^2} \frac{|Q(\eta', \vec{v}(\eta, \vec{v}; \eta'))|}{v'_\eta(\eta, \vec{v}, \eta')} \exp(-G_{\eta, \eta'}) d\eta' \\ &\leq \left\| \frac{e^{\beta\eta} Q}{\nu} \right\|_{L^\infty L^\infty_{\vartheta, \zeta}} \int_0^\eta \frac{\nu(\vec{v}'(\eta, \eta'))}{v'_\eta(\eta, \vec{v}, \eta')} \exp(\beta(\eta - \eta') - G_{\eta, \eta'}) d\eta' \\ &\leq C \left\| \frac{e^{\beta\eta} Q}{\nu} \right\|_{L^\infty L^\infty_{\vartheta, \zeta}}. \end{aligned} \quad (4.237)$$

The $v_\eta < 0$ case can be proved in a similar fashion, so we omit it here. \square

Lemma 4.11. *For any $\delta > 0$, $\vartheta > 2$ and $0 \leq \zeta \leq 1/4$, there is a constant $C(\delta)$ such that*

$$\|\mathcal{T}[Q]\|_{L^\infty L^2_\zeta} \leq C(\delta) \left\| \nu^{-1/2} Q \right\|_{L^2 L^2} + \delta \|Q\|_{L^\infty L^\infty_{\vartheta, \zeta}}. \quad (4.238)$$

Proof. We divide the proof into several cases:

Case I: For $v_\eta > 0$,

$$\mathcal{T}[Q(\eta, \vec{v})] = \int_0^\eta \frac{Q(\eta', \vec{v}(\eta, \vec{v}; \eta'))}{v'_\eta(\eta, \vec{v}, \eta')} \exp(-G_{\eta, \eta'}) d\eta'. \quad (4.239)$$

We need to estimate

$$\int_{\mathbb{R}^2} e^{2\zeta|\vec{v}|^2} \left(\int_0^\eta \frac{Q(\eta', \vec{v}(\eta, \vec{v}; \eta'))}{v'_\eta(\eta, \vec{v}, \eta')} \exp(-G_{\eta, \eta'}) d\eta' \right)^2 d\vec{v}. \quad (4.240)$$

Assume $m > 0$ is sufficiently small, $M > 0$ is sufficiently large and $\sigma > 0$ is sufficiently small which will be determined in the following. We can split the integral into the following parts

$$I = I_1 + I_2 + I_3 + I_4. \quad (4.241)$$

Case I - Type I: χ_1 : $M \leq v'_\eta(\eta, \vec{v}, \eta')$ or $M \leq v'_\phi(\eta, \vec{v}, \eta')$.

By Lemma 4.2, we have

$$|\vec{v}(\eta, \vec{v}; \eta')| + 1 \leq C\nu(\vec{v}(\eta, \vec{v}; \eta')). \quad (4.242)$$

Then for $\vartheta > 2$, since $|\vec{v}|$ is conserved along the characteristics, we have

$$\begin{aligned} I_1 &\leq C \|Q\|_{L^\infty L^\infty_{\vartheta, \zeta}}^2 \int_{\mathbb{R}^2} \chi_1 \left(\int_0^\eta \frac{1}{\langle \vec{v}' \rangle^\vartheta} \frac{\exp(-G_{\eta, \eta'})}{v'_\eta(\eta, \vec{v}, \eta')} d\eta' \right)^2 d\vec{v} \\ &\leq \frac{C}{M^\vartheta} \|Q\|_{L^\infty L^\infty_{\vartheta, \zeta}}^2 \int_{\mathbb{R}^2} \frac{1}{\langle \vec{v} \rangle^\vartheta} \left(\int_0^\eta \frac{\exp(-G_{\eta, \eta'})}{v'_\eta(\eta, \vec{v}, \eta')} d\eta' \right)^2 d\vec{v} \\ &\leq \frac{C}{M^\vartheta} \|Q\|_{L^\infty L^\infty_{\vartheta, \zeta}}^2. \end{aligned} \quad (4.243)$$

since

$$\begin{aligned} \left| \int_0^\eta \frac{\exp(-G_{\eta,\eta'})}{v'_\eta(\eta, \vec{\mathbf{v}}, \eta')} d\eta' \right| &\leq \left| \int_0^\eta \frac{-\nu(\vec{\mathbf{v}}'(\eta, \eta')) \exp(G_{\eta,\eta'})}{v'_\eta(\eta, \vec{\mathbf{v}}, \eta')} d\eta' \right| \\ &\leq \int_0^\infty e^{-y} dy = 1. \end{aligned} \quad (4.244)$$

Case I - Type II: χ_2 : $m \leq v'_\eta(\eta, \vec{\mathbf{v}}, \eta') \leq M$ and $v'_\phi(\eta, \vec{\mathbf{v}}, \eta') \leq M$.

Since along the characteristics, $|\vec{\mathbf{v}}|^2$ can be bounded by $2M^2$ and the integral domain for $\vec{\mathbf{v}}$ is finite. Then by Cauchy's inequality, we have

$$\begin{aligned} I_2 &\leq C \frac{e^{4\zeta M^2}}{m} \int_0^\eta \frac{Q^2}{\nu}(\eta', \vec{\mathbf{v}}(\eta, \vec{\mathbf{v}}; \eta')) d\eta' \int_0^\eta \frac{\nu(\vec{\mathbf{v}}'(\eta, \eta')) \exp(-2G_{\eta,\eta'})}{v'_\eta(\eta, \vec{\mathbf{v}}, \eta')} d\eta' \\ &\leq C \frac{e^{4\zeta M^2}}{m} \left\| \nu^{-1/2} Q \right\|_{L^2 L^2}^2, \end{aligned} \quad (4.245)$$

where

$$\int_0^\eta \frac{\nu(\vec{\mathbf{v}}'(\eta, \eta'))}{v'_\eta(\eta, \vec{\mathbf{v}}, \eta')} \exp(-2G_{\eta,\eta'}) d\eta' d\vec{\mathbf{v}} \leq \int_0^\infty e^{-2y} dy = \frac{1}{2}. \quad (4.246)$$

Case I - Type III: χ_3 : $0 \leq v'_\eta(\eta, \vec{\mathbf{v}}, \eta') \leq m$, $v'_\phi(\eta, \vec{\mathbf{v}}, \eta') \leq M$ and $\eta - \eta' \geq \sigma$.

In this case, we know

$$G_{\eta,\eta'} \geq \frac{\sigma}{m}. \quad (4.247)$$

Then after substitution, the integral is not from zero, but from $-\sigma/m$. Hence, we have

$$\begin{aligned} I_3 &\leq C \|Q\|_{L^\infty L^\infty_{\vec{\mathbf{v}}, \zeta}}^2 \int_{\mathbb{R}^2} \chi_3 \left(\int_0^\eta \frac{1}{\langle \vec{\mathbf{v}}' \rangle^{\vec{\mathbf{v}}}} \frac{\exp(-G_{\eta,\eta'})}{v'_\eta(\eta, \vec{\mathbf{v}}, \eta')} d\eta' \right)^2 d\vec{\mathbf{v}} \\ &\leq C \|Q\|_{L^\infty L^\infty_{\vec{\mathbf{v}}, \zeta}}^2 \int_{\mathbb{R}^2} \frac{\chi_3}{\langle \vec{\mathbf{v}} \rangle^{2\vec{\mathbf{v}}}} \left(\int_0^\eta \frac{\nu(\vec{\mathbf{v}}'(\eta, \eta')) \exp(-G_{\eta,\eta'})}{v'_\eta(\eta, \vec{\mathbf{v}}, \eta')} d\eta' \right)^2 d\vec{\mathbf{v}} \\ &\leq C \|Q\|_{L^\infty L^\infty_{\vec{\mathbf{v}}, \zeta}}^2 \left(\int_{-\sigma/m}^\infty e^{-y} dy \right)^2 \\ &\leq C e^{-\frac{\sigma}{m}} \|Q\|_{L^\infty L^\infty_{\vec{\mathbf{v}}, \zeta}}^2. \end{aligned} \quad (4.248)$$

Case I - Type IV: χ_4 : $0 \leq v'_\eta(\eta, \vec{\mathbf{v}}, \eta') \leq m$, $v'_\phi(\eta, \vec{\mathbf{v}}, \eta') \leq M$ and $\eta - \eta' \leq \sigma$.

For $\eta' \leq \eta$ and $\eta - \eta' \leq \sigma$, we have

$$v_\eta \leq C v'_\eta(\eta, \vec{\mathbf{v}}, \eta') \leq C(m + \sigma). \quad (4.249)$$

Therefore, the integral domain for v_η is very small. We have the estimate

$$\begin{aligned} I_4 &\leq C \|Q\|_{L^\infty L^\infty_{\vec{\mathbf{v}}, \zeta}} \int_{\mathbb{R}^2} \chi_4 \left(\int_0^\eta \frac{1}{\langle \vec{\mathbf{v}}' \rangle^{\vec{\mathbf{v}}}} \frac{\exp(-G_{\eta,\eta'})}{v'_\eta(\eta, \vec{\mathbf{v}}, \eta')} d\eta' \right)^2 d\vec{\mathbf{v}} \\ &\leq C \|Q\|_{L^\infty L^\infty_{\vec{\mathbf{v}}, \zeta}} \int_{\mathbb{R}^2} \frac{\chi_4}{\langle \vec{\mathbf{v}} \rangle^{\vec{\mathbf{v}}}} d\vec{\mathbf{v}} \\ &\leq C(m + \sigma) \|Q\|_{L^\infty L^\infty_{\vec{\mathbf{v}}, \zeta}}. \end{aligned} \quad (4.250)$$

Collecting all four types, we have

$$I \leq C \frac{e^{4\zeta M^2}}{m} \left\| \nu^{-1/2} Q \right\|_{L^2 L^2} + C \left(\frac{1}{M^{\vec{\mathbf{v}}}} + m + \sigma + e^{-\frac{\sigma}{m}} \right) \|Q\|_{L^\infty L^\infty_{\vec{\mathbf{v}}, \zeta}}. \quad (4.251)$$

Taking M sufficiently large, σ sufficiently small and $m \ll \sigma$, this is the desired result.

Case II:

For $v_\eta < 0$ and $|E(\eta, \vec{v})| \geq v'_\phi(\eta, \vec{v}; \infty)$,

$$\mathcal{T}[Q(\eta, \vec{v})] = \int_\eta^\infty \frac{Q(\eta', R[\vec{v}(\eta, \vec{v}; \eta')])}{v'_\eta(\eta, \vec{v}, \eta')} \exp(R[G_{\eta, \eta'}]) d\eta'. \quad (4.252)$$

We need to estimate

$$\int_{\mathbb{R}^2} e^{2\zeta|\vec{v}|^2} \left(\int_\eta^\infty \frac{Q(\eta', R[\vec{v}(\eta, \vec{v}; \eta')])}{v'_\eta(\eta, \vec{v}, \eta')} \exp(R[G_{\eta, \eta'}]) d\eta' \right)^2 d\vec{v}. \quad (4.253)$$

We can split the integral into the following types:

$$II = II_1 + II_2 + II_3. \quad (4.254)$$

Case II - Type I: χ_1 : $M \leq v'_\eta(\eta, \vec{v}, \eta')$ or $M \leq v'_\phi(\eta, \vec{v}, \eta')$.

Similar to Case I - Type I, we have

$$\begin{aligned} II_1 &\leq C \|Q\|_{L^\infty L^\infty_{\vec{\theta}, \zeta}} \int_{\mathbb{R}^2} \chi_1 \left(\int_\eta^\infty \frac{1}{\langle \vec{v}' \rangle^\vartheta} \frac{\exp(R[G_{\eta, \eta'}])}{v'_\eta(\eta, \vec{v}, \eta')} d\eta' \right)^2 d\vec{v} \\ &\leq C \frac{1}{M^\vartheta} \|Q\|_{L^\infty L^\infty_{\vec{\theta}, \zeta}}. \end{aligned} \quad (4.255)$$

Case II - Type II: χ_2 : $m \leq v'_\eta(\eta, \vec{v}, \eta') \leq M$ and $v'_\phi(\eta, \vec{v}, \eta') \leq M$.

Similar to Case I - Type II, by Cauchy's inequality, we have

$$\begin{aligned} II_2 &\leq C \frac{e^{4\zeta M^2}}{m} \int_\eta^\infty \frac{Q^2}{\nu}(\eta', \vec{v}(\eta, \vec{v}; \eta')) d\eta' \int_\eta^\infty \frac{\exp(2R[G_{\eta, \eta'}])}{v'_\eta(\eta, \vec{v}, \eta')} d\eta' \\ &\leq C \frac{e^{4\zeta M^2}}{m} \left\| \nu^{-1/2} Q \right\|_{L^2 L^2}. \end{aligned} \quad (4.256)$$

Case I - Type III: χ_3 : $0 \leq v'_\eta(\eta, \vec{v}, \eta') \leq m$ and $v'_\phi(\eta, \vec{v}, \eta') \leq M$.

In this case, we can directly verify the fact

$$v_\eta \leq v'_\eta(\eta, \vec{v}, \eta') \quad (4.257)$$

for $\eta \leq \eta'$. Then we know the integral of v_η is always in a small domain. Similar to Case I - Type IV, we have the estimate

$$II_3 \leq Cm \|Q\|_{L^\infty L^\infty_{\vec{\theta}, \zeta}}. \quad (4.258)$$

Hence, collecting all three types, we obtain

$$II \leq C \frac{e^{4\zeta M^2}}{m} \left\| \nu^{-1/2} Q \right\|_{L^2 L^2} + C \left(\frac{1}{M^\vartheta} + m \right) \|Q\|_{L^\infty L^\infty_{\vec{\theta}, \zeta}}. \quad (4.259)$$

Taking M sufficiently large and m sufficiently small, this is the desired result.

Case III:

For $v_\eta < 0$ and $|E(\eta, \vec{v})| \leq v'_\phi(\eta, \vec{v}; \infty)$,

$$(4.260)$$

$$\begin{aligned} \mathcal{T}[Q(\eta, \vec{v})] &= \left(\int_0^{\eta^+} \frac{Q(\eta', \vec{v}(\eta, \vec{v}; \eta'))}{\nu(\vec{v}(\eta, \vec{v}; \eta'))} \exp(-G_{\eta^+, \eta'} - R[G_{\eta^+, \eta}]) d\eta' \right. \\ &\quad \left. + \int_\eta^{\eta^+} \frac{Q(\eta', R[\vec{v}(\eta, \vec{v}; \eta')])}{\nu(\vec{v}(\eta, \vec{v}; \eta'))} \exp(R[G_{\eta, \eta'}]) d\eta' \right). \end{aligned}$$

This is a combination of Case I and Case II, so it naturally holds. \square

4.2.4. Estimates of ϵ -Milne problem.

Lemma 4.12. Assume (4.7) and (4.8) hold. The solution $g(\eta, \vec{v})$ to the ϵ -Milne problem (4.1) satisfies for $\vartheta > 2$ and $0 \leq \zeta \leq 1/4$,

$$\|g - g_\infty\|_{L^\infty L^\infty_{\vartheta, \zeta}} \leq C + C\|g - g_\infty\|_{L^2 L^2}. \quad (4.261)$$

Proof. Define $u = g - g_\infty$. Then u satisfies the equation

$$\begin{cases} v_\eta \frac{\partial u}{\partial \eta} + G(\eta) \left(v_\phi^2 \frac{\partial u}{\partial v_\eta} - v_\eta v_\phi \frac{\partial u}{\partial v_\phi} \right) + \mathcal{L}[u] &= S(\eta, \vec{v}) + g_{2, \infty} G(\eta) \sqrt{\mu} v_\eta v_\phi = \tilde{S}, \\ u(0, \vec{v}) &= (h - g_\infty)(\vec{v}) = p(\vec{v}) \text{ for } v_\eta > 0, \\ \int_{\mathbb{R}^2} v_\eta \sqrt{\mu} u(0, \vec{v}) d\vec{v} &= m_f[g] - g_{1, \infty}, \\ \lim_{\eta \rightarrow \infty} u(\eta, \vec{v}) &= 0, \end{cases} \quad (4.262)$$

Since $u = \mathcal{A}[p] + \mathcal{T}[K[u] + \tilde{S}]$, based on Lemma 4.11, we have

$$\begin{aligned} \|u - \mathcal{A}[p]\|_{L^\infty L^\infty_\zeta} &= \|\mathcal{T}[K[u] + \tilde{S}]\|_{L^\infty L^2} \\ &\leq C(\delta) \left(\|\nu^{-1/2} K[u]\|_{L^2 L^2} + \|\nu^{-1/2} \tilde{S}\|_{L^2 L^2} \right) + \delta \left(\|K[u]\|_{L^\infty L^\infty_{\vartheta, \zeta}} + \|\tilde{S}\|_{L^\infty L^\infty_{\vartheta, \zeta}} \right) \\ &\leq C(\delta) \left(\|u\|_{L^2 L^2} + \|\tilde{S}\|_{L^2 L^2} \right) + \delta \left(\|K[u]\|_{L^\infty L^\infty_{\vartheta, \zeta}} + \|\tilde{S}\|_{L^\infty L^\infty_{\vartheta, \zeta}} \right), \end{aligned} \quad (4.263)$$

where we can directly verify

$$\|\nu^{-1/2} K[u]\|_{L^2 L^2} \leq \|u\|_{L^2 L^2}, \quad (4.264)$$

$$\|\nu^{-1/2} \tilde{S}\|_{L^2 L^2} \leq \|\tilde{S}\|_{L^2 L^2}. \quad (4.265)$$

In [10, Lemma 3.3.1], it is shown that

$$\|K[u]\|_{L^\infty L^\infty_{\vartheta, \zeta}} \leq \|u\|_{L^\infty L^\infty_{\vartheta-1, \zeta}}, \quad (4.266)$$

$$\|K[u]\|_{L^\infty L^\infty_{0, \zeta}} \leq \|u\|_{L^\infty L^\infty_\zeta}. \quad (4.267)$$

Since $u = \mathcal{A}[p] + \mathcal{T}[K[u] + \tilde{S}]$, for ϵ and δ sufficiently small, we can estimate

$$\begin{aligned} \|u\|_{L^\infty L^\infty_{\vartheta, \zeta}} &\leq C \left(\|\mathcal{T}[K[u]]\|_{L^\infty L^\infty_{\vartheta, \zeta}} + \|\mathcal{T}[\tilde{S}]\|_{L^\infty L^\infty_{\vartheta, \zeta}} + \|\mathcal{A}[p]\|_{L^\infty_{\vartheta, \zeta}} \right) \\ &\leq C \left(\|K[u]\|_{L^\infty L^\infty_{\vartheta, \zeta}} + \|S\|_{L^\infty L^\infty_{\vartheta, \zeta}} + \|\mathcal{A}[p]\|_{L^\infty_{\vartheta, \zeta}} \right) \\ &\leq C \left(\|u\|_{L^\infty L^\infty_{\vartheta-1, \zeta}} + \|S\|_{L^\infty L^\infty_{\vartheta, \zeta}} + \|\mathcal{A}[p]\|_{L^\infty_{\vartheta, \zeta}} \right) \\ &\leq \dots \\ &\leq C \left(\|K[u]\|_{L^\infty L^\infty_{0, \zeta}} + \|S\|_{L^\infty L^\infty_{\vartheta, \zeta}} + \|\mathcal{A}[p]\|_{L^\infty_{\vartheta, \zeta}} \right) \\ &\leq C \left(\|u\|_{L^\infty L^\infty_\zeta} + \|S\|_{L^\infty L^\infty_{\vartheta, \zeta}} + \|\mathcal{A}[p]\|_{L^\infty_{\vartheta, \zeta}} \right) \\ &\leq C(\delta) \left(\|\nu^{-1/2} K[u]\|_{L^2 L^2} + \|\nu^{-1/2} \tilde{S}\|_{L^2 L^2} \right) + \delta \left(\|K[u]\|_{L^\infty L^\infty_{\vartheta, \zeta}} + \|\tilde{S}\|_{L^\infty L^\infty_{\vartheta, \zeta}} \right) \\ &\quad + C \left(\|S\|_{L^\infty L^\infty_{\vartheta, \zeta}} + \|\mathcal{A}[p]\|_{L^\infty_{\vartheta, \zeta}} \right). \end{aligned} \quad (4.268)$$

Therefore, absorbing $\delta \|K[u]\|_{L^\infty L^\infty_{\vartheta,\zeta}}$ into the right-hand side of the second inequality implies

$$\|K[u]\|_{L^\infty L^\infty_{\vartheta,\zeta}} \leq C \left(\left\| \nu^{-1/2} K[u] \right\|_{L^2 L^2} + \|S\|_{L^\infty L^\infty_{\vartheta,\zeta}} + \|p\|_{L^\infty_{\vartheta,\zeta}} \right) \quad (4.269)$$

Therefore, we have

$$\begin{aligned} \|u\|_{L^\infty L^\infty_{\vartheta,\zeta}} &\leq C \left(\|K[u]\|_{L^\infty L^\infty_{\vartheta,\zeta}} + \|S\|_{L^\infty L^\infty_{\vartheta,\zeta}} + \|\mathcal{A}[p]\|_{L^\infty_{\vartheta,\zeta}} \right) \\ &\leq C \left(\|u\|_{L^2 L^2} + \|S\|_{L^\infty L^\infty_{\vartheta,\zeta}} + \|p\|_{L^\infty_{\vartheta,\zeta}} \right). \end{aligned}$$

Then our result naturally follows. \square

Lemma 4.13. *Assume (4.7) and (4.8) hold. There exists a unique solution $g(\eta, \vec{v})$ to the ϵ -Milne problem (4.1) satisfying for $\vartheta > 2$ and $0 \leq \zeta \leq 1/4$,*

$$\|g - g_\infty\|_{L^\infty L^\infty_{\vartheta,\zeta}} \leq C. \quad (4.270)$$

Proof. Based on Lemma 4.6 and Lemma 4.12, this is obvious. \square

Theorem 4.14. *Assume (4.7) and (4.8) hold. There exists a unique solution $\mathcal{G}(\eta, \vec{v})$ to the ϵ -Milne problem (4.11) satisfying for $\vartheta > 2$ and $0 \leq \zeta \leq 1/4$,*

$$\|\mathcal{G}\|_{L^\infty L^\infty_{\vartheta,\zeta}} \leq C. \quad (4.271)$$

Proof. Based on Theorem 4.7 and Lemma 4.13, this is obvious. \square

4.3. Exponential Decay.

Theorem 4.15. *Assume (4.7) and (4.8) hold. For sufficiently small K_0 , there exists a unique solution $\mathcal{G}(\eta, \vec{v})$ to the ϵ -Milne problem (4.11) satisfying for $\vartheta > 2$ and $0 \leq \zeta \leq 1/4$,*

$$\|e^{K_0 \eta} \mathcal{G}\|_{L^\infty L^\infty_{\vartheta,\zeta}} \leq C. \quad (4.272)$$

Proof. Define $U = e^{K_0 \eta} \mathcal{G}$. Then U satisfies the equation

$$\left\{ \begin{array}{l} v_\eta \frac{\partial U}{\partial \eta} + G(\eta) \left(v_\phi^2 \frac{\partial U}{\partial v_\eta} - v_\eta v_\phi \frac{\partial U}{\partial v_\phi} \right) + \mathcal{L}[U] = e^{K_0 \eta} S(\eta, \vec{v}) + K_0 v_\eta U, \\ U(0, \vec{v}) = e^{K_0 \eta} (h - \tilde{h})(\vec{v}) \text{ for } v_\eta > 0, \\ \int_{\mathbb{R}^2} v_\eta \sqrt{\mu} U(0, \vec{v}) d\vec{v} = m_f[g] - \int_{\mathbb{R}^2} \sqrt{\mu} \tilde{h}(\vec{v}) d\vec{v}, \\ \lim_{\eta \rightarrow \infty} U(\eta, \vec{v}) = 0, \end{array} \right. \quad (4.273)$$

We divide the proof into several steps:

Step 1: L^2 Estimates for $S = 0$ and $m_f[U] = 0$.

In the proof of Lemma 4.4, we already show

$$\int_0^\infty e^{2K_0 \eta} \langle w_{\mathcal{G}}, w_{\mathcal{G}} \rangle(\eta) d\eta \leq C. \quad (4.274)$$

We can decompose $\mathcal{G} = w_{\mathcal{G}} + q_{\mathcal{G}}$. Since $\lim_{\eta \rightarrow \infty} \mathcal{G}(\eta, \vec{v}) = 0$, we naturally have $q_{\mathcal{G}} = 0$. Then using the orthogonal relation and zero mass-flux, we have

$$\int_0^\infty e^{2K_0 \eta} \int_{\mathbb{R}^2} \mathcal{G}^2(\eta, \vec{v}) d\vec{v} d\eta = \int_0^\infty e^{2K_0 \eta} \langle q_{\mathcal{G}}, q_{\mathcal{G}} \rangle(\eta) d\eta + \int_0^\infty e^{2K_0 \eta} \langle w_{\mathcal{G}}, w_{\mathcal{G}} \rangle(\eta) d\eta. \quad (4.275)$$

Similar to Step 4 in the proof of Lemma 4.4, using the exponential decay of $w_{\mathcal{G}}$, we have

$$\int_0^\infty e^{2K_0 \eta} \langle q_{\mathcal{G}}, q_{\mathcal{G}} \rangle(\eta) d\eta \leq C \int_0^\infty e^{2K_0 \eta} \langle w_{\mathcal{G}}, w_{\mathcal{G}} \rangle(\eta) d\eta \quad (4.276)$$

This shows

$$\|U\|_{L^2 L^2} < C. \quad (4.277)$$

Step 2: L^2 Estimates for general source term and mass flux.

We follow the idea in the proof of Lemma 4.5. Note that all the auxiliary functions we construct decays exponentially. Hence, the result naturally follows.

Step 3: L^∞ Estimates.

By a similar argument of Lemma 4.12, Since $u = \mathcal{A}[p] + \mathcal{T}[K[u] + \tilde{S}]$, similar to the proof of Lemma 4.12, we have

$$\|e^{K_0\eta}\mathcal{G}\|_{L^\infty L^\infty_{\vartheta,\zeta}} \leq C \left(\|e^{K_0\eta}\mathcal{G}\|_{L^2 L^2} + \|e^{K_0\eta}S\|_{L^\infty L^\infty_{\vartheta,\zeta}} + \|p\|_{L^\infty_{\vartheta,\zeta}} \right) \quad (4.278)$$

Then we naturally obtain the result. \square

Our results can also be applied to the Milne problem without geometric correction.

Remark 4.16. Taking $G = 0$, we consider the Milne problem

$$\begin{cases} v_\eta \frac{\partial g}{\partial \eta} + \mathcal{L}[g] = S(\eta, \phi, \vec{\mathbf{v}}), \\ g(0, \phi, \vec{\mathbf{v}}) = h(\phi, \vec{\mathbf{v}}) \text{ for } v_\eta > 0, \\ \int_{\mathbb{R}^2} v_\eta \sqrt{\mu} g(0, \phi, \vec{\mathbf{v}}) d\vec{\mathbf{v}} = m_f[g], \\ \lim_{\eta \rightarrow \infty} g(\eta, \phi, \vec{\mathbf{v}}) = g_\infty(\phi, \vec{\mathbf{v}}). \end{cases} \quad (4.279)$$

Then there exists

$$\tilde{h}(\phi, \vec{\mathbf{v}}) = \tilde{D}_0(\phi)\psi_0 + \tilde{D}_1(\phi)\psi_1 + \tilde{D}_2(\phi)\psi_2 + \tilde{D}_3(\phi)\psi_3, \quad (4.280)$$

such that the Milne problem for $\mathcal{G}(\eta, \phi, \vec{\mathbf{v}})$ in the domain $(\eta, \phi, \vec{\mathbf{v}}) \in [0, \infty) \times [-\pi, \pi) \times \mathbb{R}^2$

$$\begin{cases} v_\eta \frac{\partial \mathcal{G}}{\partial \eta} + \mathcal{L}[\mathcal{G}] = S(\eta, \phi, \vec{\mathbf{v}}), \\ \mathcal{G}(0, \phi, \vec{\mathbf{v}}) = h(\phi, \vec{\mathbf{v}}) - \tilde{h}(\phi, \vec{\mathbf{v}}) \text{ for } v_\eta > 0, \\ \int_{\mathbb{R}^2} v_\eta \sqrt{\mu} \mathcal{G}(0, \phi, \vec{\mathbf{v}}) d\vec{\mathbf{v}} = m_f[g] - \int_{\mathbb{R}^2} v_\eta \sqrt{\mu} \tilde{h}(\phi, \vec{\mathbf{v}}) d\vec{\mathbf{v}}, \\ \lim_{\eta \rightarrow \infty} \mathcal{G}(\eta, \phi, \vec{\mathbf{v}}) = 0. \end{cases} \quad (4.281)$$

is well-posed in L^∞ and decays exponentially.

5. DIFFUSIVE LIMIT AND WELL-POSEDNESS

We prove the diffusive limit and well-posedness of the Boltzmann equation (1.12).

Theorem 5.1. For given $B^\epsilon > 0$ satisfying (1.10) and $0 < \epsilon < 1$, there exists a unique positive solution $F^\epsilon = \mu + \sqrt{\mu} f^\epsilon$ to the stationary Boltzmann equation (1.1), where

$$f^\epsilon = \epsilon^3 R_N + \left(\sum_{k=1}^N \epsilon^k \mathcal{F}_k^\epsilon \right) + \left(\sum_{k=1}^N \epsilon^k \mathcal{F}_k^\epsilon \right), \quad (5.1)$$

for $N \geq 3$ and R_N satisfies

$$\begin{cases} \epsilon \vec{v} \cdot \nabla_x R_N + \mathcal{L}[R_N] = \epsilon^3 \Gamma[R_N, R_N] + 2\Gamma[R_N, \mathcal{Q}_N + \mathcal{Q}_N] + S_N \text{ in } \Omega, \\ R_N(\vec{x}_0, \vec{v}) = h_N \text{ for } \vec{v} \cdot \vec{n} < 0 \text{ and } \vec{x}_0 \in \partial\Omega, \end{cases} \quad (5.2)$$

with

$$\begin{aligned} S_N = & - \sum_{1 \leq i, j \leq N}^{i+j \geq N+1} \epsilon^{i+j-3} \Gamma[\mathcal{F}_i^\epsilon, \mathcal{F}_j^\epsilon] - \Upsilon_0 \sum_{1 \leq i, j \leq N}^{i+j \geq N+1} \epsilon^{i+j-3} \left(\Gamma[\mathcal{F}_i^\epsilon, \mathcal{F}_j^\epsilon] + 2\Gamma[\mathcal{F}_i^\epsilon, \mathcal{F}_j^\epsilon] \right) \\ & - \epsilon^{N-2} \vec{v} \cdot \nabla_x \mathcal{F}_N^\epsilon - \sum_{k=1}^N \epsilon^{k-3} v_\eta \frac{\partial \Upsilon_0}{\partial \eta} \mathcal{G}_k^\epsilon + \epsilon^{N-2} \frac{\Upsilon_0}{1 - \epsilon\eta} v_\phi \frac{\partial \mathcal{G}_N^\epsilon}{\partial \phi}, \end{aligned}$$

and

$$h_N = \sum_{k=N+1}^{\infty} \epsilon^{k-3} b_k,$$

\mathcal{F}_k^ϵ and \mathcal{F}_k^ϵ satisfy (3.60) and (3.53). Also, there exists a $C > 0$ such that f^ϵ satisfies

$$\left\| \langle \vec{v} \rangle^\vartheta e^{\zeta |\vec{v}|^2} f^\epsilon \right\|_{L^\infty} \leq C\epsilon, \quad (5.3)$$

for any $\vartheta > 2$ and $0 \leq \zeta \leq 1/4$

.

Proof. We divide the proof into several steps:

Step 1: Remainder definitions.

We combine the interior solution and boundary layer as follows:

$$f^\epsilon \sim \sum_{k=1}^{\infty} \epsilon^k \mathcal{F}_k^\epsilon + \sum_{k=1}^{\infty} \epsilon^k \mathcal{F}_k^\epsilon. \quad (5.4)$$

Define the remainder as

$$R_N = \frac{1}{\epsilon^3} \left(f^\epsilon - \sum_{k=1}^N \epsilon^k \mathcal{F}_k^\epsilon - \sum_{k=1}^N \epsilon^k \mathcal{F}_k^\epsilon \right) = \frac{1}{\epsilon^3} \left(f^\epsilon - \mathcal{Q}_N - \mathcal{Q}_N \right), \quad (5.5)$$

where

$$\mathcal{Q}_N = \sum_{k=1}^N \epsilon^k \mathcal{F}_k^\epsilon, \quad (5.6)$$

$$\mathcal{Q}_N = \sum_{k=1}^N \epsilon^k \mathcal{F}_k^\epsilon. \quad (5.7)$$

Noting the equation (3.37) is equivalent to the equation (1.12), we write \mathcal{L} to denote the linearized Boltzmann operator as follows:

$$\begin{aligned} \mathcal{L}[f] &= \epsilon \vec{v} \cdot \nabla_x u + \mathcal{L}[f] \\ &= v_\eta \frac{\partial f}{\partial \eta} - \frac{\epsilon}{1 - \epsilon \eta} \left(v_\phi \frac{\partial f}{\partial \phi} + v_\phi^2 \frac{\partial f}{\partial v_\eta} - v_\eta v_\phi \frac{\partial f}{\partial v_\phi} \right) + \mathcal{L}[f]. \end{aligned} \quad (5.8)$$

Step 2: Estimates of $\mathcal{L}[R_N]$.

The interior contribution can be estimated as

$$\begin{aligned} \mathcal{L}[\mathcal{Q}_N] &= \epsilon \vec{v} \cdot \nabla_x \mathcal{Q}_N + \mathcal{L}[\mathcal{Q}_N] = \sum_{k=1}^N \epsilon^k \left(\epsilon \vec{v} \cdot \nabla_x \mathcal{F}_k^\epsilon + \mathcal{L}[\mathcal{F}_k^\epsilon] \right) \\ &= \mathcal{L}[\mathcal{F}_1^\epsilon] + \sum_{k=2}^N \epsilon^k \left(\vec{v} \cdot \nabla_x \mathcal{F}_{k-1}^\epsilon + \mathcal{L}[\mathcal{F}_k^\epsilon] \right) + \epsilon^{N+1} \vec{v} \cdot \nabla_x \mathcal{F}_N^\epsilon \\ &= \epsilon^{N+1} \vec{v} \cdot \nabla_x \mathcal{F}_N^\epsilon + \sum_{1 \leq i, j \leq N}^{i+j \leq N} \epsilon^{i+j} \Gamma[\mathcal{F}_i^\epsilon, \mathcal{F}_j^\epsilon]. \end{aligned} \quad (5.9)$$

The boundary layer is $\mathcal{F}_k^\epsilon = \mathcal{G}_k^\epsilon \cdot \Upsilon_0$ where \mathcal{G}_k^ϵ solves the ϵ -Milne problem. Notice $\Upsilon_0 \Upsilon = \Upsilon_0$, so the boundary layer contribution can be estimated as

$$\begin{aligned}
\mathcal{L}[\mathcal{Q}_N] &= v_\eta \frac{\partial \mathcal{Q}_N}{\partial \eta} - \frac{\epsilon}{1 - \epsilon \eta} \left(v_\phi \frac{\partial \mathcal{Q}_N}{\partial \phi} + v_\phi^2 \frac{\partial \mathcal{Q}_N}{\partial v_\eta} - v_\eta v_\phi \frac{\partial \mathcal{Q}_N}{\partial v_\phi} \right) + \mathcal{L}[\mathcal{Q}_N] \\
&= \sum_{k=1}^N \epsilon^k \left(v_\eta \frac{\partial \mathcal{F}_k^\epsilon}{\partial \eta} - \frac{\epsilon}{1 - \epsilon \eta} \left(v_\phi \frac{\partial \mathcal{F}_k^\epsilon}{\partial \phi} + v_\phi^2 \frac{\partial \mathcal{F}_k^\epsilon}{\partial v_\eta} - v_\eta v_\phi \frac{\partial \mathcal{F}_k^\epsilon}{\partial v_\phi} \right) + \mathcal{L}[\mathcal{F}_k^\epsilon] \right) \\
&= \sum_{k=1}^N \epsilon^k \left(v_\eta \frac{\partial \mathcal{G}_k^\epsilon}{\partial \eta} \Upsilon_0 - \frac{\epsilon \Upsilon_0}{1 - \epsilon \eta} \left(v_\phi \frac{\partial \mathcal{G}_k^\epsilon}{\partial \phi} + v_\phi^2 \frac{\partial \mathcal{G}_k^\epsilon}{\partial v_\eta} - v_\eta v_\phi \frac{\partial \mathcal{G}_k^\epsilon}{\partial v_\phi} \right) + \Upsilon_0 \mathcal{L}[\mathcal{G}_k^\epsilon] \right) \\
&\quad + \sum_{k=1}^N \epsilon^k v_\eta \frac{\partial \Upsilon_0}{\partial \eta} \mathcal{G}_k^\epsilon \\
&= \sum_{k=1}^N \epsilon^k \left(v_\eta \frac{\partial \mathcal{G}_k^\epsilon}{\partial \eta} \Upsilon_0 - \frac{\epsilon \Upsilon_0 \Upsilon}{1 - \epsilon \eta} \left(v_\phi \frac{\partial \mathcal{G}_k^\epsilon}{\partial \phi} + v_\phi^2 \frac{\partial \mathcal{G}_k^\epsilon}{\partial v_\eta} - v_\eta v_\phi \frac{\partial \mathcal{G}_k^\epsilon}{\partial v_\phi} \right) + \Upsilon_0 \mathcal{L}[\mathcal{G}_k^\epsilon] \right) \\
&\quad + \sum_{k=1}^N \epsilon^k v_\eta \frac{\partial \Upsilon_0}{\partial \eta} \mathcal{G}_k^\epsilon \\
&= \Upsilon_0 \sum_{k=1}^N \epsilon^k \left(v_\eta \frac{\partial \mathcal{G}_k^\epsilon}{\partial \eta} - \frac{\epsilon \Upsilon}{1 - \epsilon \eta} \left(v_\phi^2 \frac{\partial \mathcal{G}_k^\epsilon}{\partial v_\eta} - v_\eta v_\phi \frac{\partial \mathcal{G}_k^\epsilon}{\partial v_\phi} \right) + \mathcal{L}[\mathcal{G}_k^\epsilon] \right) \\
&\quad + \sum_{k=1}^N \epsilon^k v_\eta \frac{\partial \Upsilon_0}{\partial \eta} \mathcal{G}_k^\epsilon - \sum_{k=1}^N \epsilon^{k+1} \frac{\Upsilon_0}{1 - \epsilon \eta} v_\phi \frac{\partial \mathcal{G}_k^\epsilon}{\partial \phi} \\
&= \sum_{k=1}^N \epsilon^k v_\eta \frac{\partial \Upsilon_0}{\partial \eta} \mathcal{G}_k^\epsilon - \epsilon^{N+1} \frac{\Upsilon_0}{1 - \epsilon \eta} v_\phi \frac{\partial \mathcal{G}_N^\epsilon}{\partial \phi} + \Upsilon_0 \sum_{1 \leq i, j \leq N} \epsilon^{i+j} \left(\Gamma[\mathcal{F}_i^\epsilon, \mathcal{F}_j^\epsilon] + 2\Gamma[\mathcal{F}_i^\epsilon, \mathcal{F}_j^\epsilon] \right).
\end{aligned} \tag{5.10}$$

Note that for any $f, g \in L^2$,

$$\mathbb{P}[\Gamma(f, g)] = 0. \tag{5.11}$$

Since

$$\mathcal{L}[f^\epsilon] = \Gamma[f^\epsilon, f^\epsilon], \tag{5.12}$$

then we can naturally obtain

$$\begin{aligned}
\mathcal{L}[R_N] &= \frac{1}{\epsilon^3} \mathcal{L}[f^\epsilon - \mathcal{Q}_N - \mathcal{Q}_N] = \frac{1}{\epsilon^3} \mathcal{L}[f^\epsilon] - \frac{1}{\epsilon^3} \mathcal{L}[\mathcal{Q}_N] - \frac{1}{\epsilon^3} \mathcal{L}[\mathcal{Q}_N] \\
&= \frac{1}{\epsilon^3} \Gamma[\mathcal{Q}_N + \mathcal{Q}_N + \epsilon^3 R_N, \mathcal{Q}_N + \mathcal{Q}_N + \epsilon^3 R_N] \\
&\quad - \sum_{1 \leq i, j \leq N} \epsilon^{i+j-3} \Gamma[\mathcal{F}_i^\epsilon, \mathcal{F}_j^\epsilon] - \psi_0 \sum_{1 \leq i, j \leq N} \epsilon^{i+j-3} \left(\Gamma[\mathcal{F}_i^\epsilon, \mathcal{F}_j^\epsilon] + 2\Gamma[\mathcal{F}_i^\epsilon, \mathcal{F}_j^\epsilon] \right) \\
&\quad - \epsilon^{N-2} \vec{v} \cdot \nabla_x \mathcal{F}_N^\epsilon - \sum_{k=1}^N \epsilon^{k-3} v_\eta \frac{\partial \Upsilon_0}{\partial \eta} \mathcal{G}_k^\epsilon + \epsilon^{N-2} \frac{\Upsilon_0}{1 - \epsilon \eta} v_\phi \frac{\partial \mathcal{G}_N^\epsilon}{\partial \phi} \\
&= \epsilon^3 \Gamma[R_N, R_N] + 2\Gamma[R_N, \mathcal{Q}_N + \mathcal{Q}_N] \\
&\quad - \sum_{1 \leq i, j \leq N} \epsilon^{i+j-3} \Gamma[\mathcal{F}_i^\epsilon, \mathcal{F}_j^\epsilon] - \psi_0 \sum_{1 \leq i, j \leq N} \epsilon^{i+j-3} \left(\Gamma[\mathcal{F}_i^\epsilon, \mathcal{F}_j^\epsilon] + 2\Gamma[\mathcal{F}_i^\epsilon, \mathcal{F}_j^\epsilon] \right) \\
&\quad - \epsilon^{N-2} \vec{v} \cdot \nabla_x \mathcal{F}_N^\epsilon - \sum_{k=1}^N \epsilon^{k-3} v_\eta \frac{\partial \Upsilon_0}{\partial \eta} \mathcal{G}_k^\epsilon + \epsilon^{N-2} \frac{\Upsilon_0}{1 - \epsilon \eta} v_\phi \frac{\partial \mathcal{G}_N^\epsilon}{\partial \phi}.
\end{aligned} \tag{5.13}$$

Step 3: Estimates of R_N .

R_N satisfies the equation

$$\begin{cases} \epsilon \vec{v} \cdot \nabla_x R_N + \mathcal{L}[R_N] &= \epsilon^3 \Gamma[R_N, R_N] + 2\Gamma[R_N, \mathcal{Q}_N + \mathcal{Q}_N] + S_N \quad \text{in } \Omega, \\ R_N(\vec{x}_0, \vec{v}) &= h_N \quad \text{for } \vec{v} \cdot \vec{n} < 0 \quad \text{and } \vec{x}_0 \in \partial\Omega, \end{cases} \quad (5.14)$$

where

$$\begin{aligned} S_N &= - \sum_{1 \leq i, j \leq N}^{i+j \geq N+1} \epsilon^{i+j-3} \Gamma[\mathcal{F}_i^\epsilon, \mathcal{F}_j^\epsilon] - \psi_0 \sum_{1 \leq i, j \leq N}^{i+j \geq N+1} \epsilon^{i+j-3} \left(\Gamma[\mathcal{F}_i^\epsilon, \mathcal{F}_j^\epsilon] + 2\Gamma[\mathcal{F}_i^\epsilon, \mathcal{F}_j^\epsilon] \right) \\ &\quad - \epsilon^{N-2} \vec{v} \cdot \nabla_x \mathcal{F}_N^\epsilon - \sum_{k=1}^N \epsilon^{k-3} v_\eta \frac{\partial \Upsilon_0}{\partial \eta} \mathcal{G}_k^\epsilon + \epsilon^{N-2} \frac{\Upsilon_0}{1-\epsilon\eta} v_\phi \frac{\partial \mathcal{G}_N^\epsilon}{\partial \phi}, \end{aligned}$$

and

$$h_N = \sum_{k=N+1}^{\infty} \epsilon^{k-3} b_k.$$

By the classical estimate of two-dimensional Stokes-Fourier equations, exponential decay of \mathcal{F}_k^ϵ and (5.11), we can directly verify

$$\left\| \langle \vec{v} \rangle^\vartheta e^{\zeta |\vec{v}|^2} S_N \right\|_{L^\infty} \leq C \epsilon^{N-2}, \quad (5.15)$$

$$\left| \langle \vec{v} \rangle^\vartheta e^{\zeta |\vec{v}|^2} h_N \right|_{L^\infty} \leq C \epsilon^{N-2}. \quad (5.16)$$

Based on Theorem 2.5, we have

$$\|R_N\|_{L^2} + |R_N|_{L^2_+} \leq C \left(\frac{1}{\epsilon^2} \|S_N\|_{L^2} + \frac{1}{\epsilon} \|\epsilon^3 \Gamma[R_N, R_N]\|_{L^2} + \frac{1}{\epsilon} \|2\Gamma[R_N, \mathcal{Q}_N + \mathcal{Q}_N]\|_{L^2} + \frac{1}{\epsilon^{1/2}} |h_N|_{L^2_-} \right). \quad (5.17)$$

Then since

$$\left| \langle \vec{v} \rangle^\vartheta e^{\zeta |\vec{v}|^2} (B^\epsilon - \mu) \right|_{L^\infty} \leq C_0 \epsilon, \quad (5.18)$$

for C_0 is sufficiently small, we deduce b_1 is sufficiently small. Hence, small boundary data naturally yields

$$\left\| \langle \vec{v} \rangle^\vartheta e^{\zeta |\vec{v}|^2} (\mathcal{Q}_N + \mathcal{Q}_N) \right\|_{L^\infty} \leq \delta \epsilon, \quad (5.19)$$

which further implies

$$\frac{1}{\epsilon} \|2\Gamma[R_N, \mathcal{Q}_N + \mathcal{Q}_N]\|_{L^2} \leq \frac{C}{\epsilon} \|R_N\|_{L^2} \left\| \langle \vec{v} \rangle^\vartheta e^{\zeta |\vec{v}|^2} (\mathcal{Q}_N + \mathcal{Q}_N) \right\|_{L^\infty} \leq \delta \|R_N\|_{L^2}, \quad (5.20)$$

for some small $\delta > 0$. Hence, absorbing them into the left-hand side of (5.17) yields

$$\|R_N\|_{L^2} + |R_N|_{L^2_+} \leq C \left(\frac{1}{\epsilon^2} \|S_N\|_{L^2} + \epsilon \|\Gamma[R_N, R_N]\|_{L^2} + \frac{1}{\epsilon^{1/2}} |h_N|_{L^2_-} \right). \quad (5.21)$$

Based on Theorem 2.7, we have

$$\begin{aligned}
& \left\| \langle \vec{v} \rangle^\vartheta e^{\zeta |\vec{v}|^2} R_N \right\|_{L^\infty} + \left| \langle \vec{v} \rangle^\vartheta e^{\zeta |\vec{v}|^2} R_N \right|_{L^\infty_+} \\
& \leq C \left(\frac{1}{\epsilon} \|R_N\|_{L^2} + \left\| \langle \vec{v} \rangle^\vartheta e^{\zeta |\vec{v}|^2} S_N \right\|_{L^\infty} + \left| \langle \vec{v} \rangle^\vartheta e^{\zeta |\vec{v}|^2} h_N \right|_{L^\infty_-} \right. \\
& \quad \left. + \left\| \langle \vec{v} \rangle^\vartheta e^{\zeta |\vec{v}|^2} \epsilon^2 \Gamma[R_N, R_N] \right\|_{L^\infty} + \left\| 2 \langle \vec{v} \rangle^\vartheta e^{\zeta |\vec{v}|^2} \Gamma[R_N, \mathcal{Q}_N + \mathcal{Q}_N] \right\|_{L^\infty} \right) \\
& \leq C \left(\epsilon \|\Gamma[R_N, R_N]\|_{L^2} + \frac{1}{\epsilon^3} \|S_N\|_{L^2} + \frac{1}{\epsilon^{3/2}} |h_N|_{L^2_-} + \left\| \langle \vec{v} \rangle^\vartheta e^{\zeta |\vec{v}|^2} S_N \right\|_{L^\infty} + \left| \langle \vec{v} \rangle^\vartheta e^{\zeta |\vec{v}|^2} h_N \right|_{L^\infty_-} \right. \\
& \quad \left. + \epsilon^3 \left\| \langle \vec{v} \rangle^\vartheta e^{\zeta |\vec{v}|^2} \Gamma[R_N, R_N] \right\|_{L^\infty} + \left\| 2 \langle \vec{v} \rangle^\vartheta e^{\zeta |\vec{v}|^2} \Gamma[R_N, \mathcal{Q}_N + \mathcal{Q}_N] \right\|_{L^\infty} \right) \\
& \leq C \left(\epsilon \|\Gamma[R_N, R_N]\|_{L^2} + \epsilon^3 \left\| \langle \vec{v} \rangle^\vartheta e^{\zeta |\vec{v}|^2} \Gamma[R_N, R_N] \right\|_{L^\infty} + \epsilon \left\| \langle \vec{v} \rangle^\vartheta e^{\zeta |\vec{v}|^2} R_N \right\|_{L^\infty} + \epsilon^{N-5} \right).
\end{aligned} \tag{5.22}$$

Moreover, we can directly estimate

$$\|\epsilon \Gamma[R_N, R_N]\|_{L^2} \leq C \epsilon \left\| \langle \vec{v} \rangle^\vartheta e^{\zeta |\vec{v}|^2} R_N \right\|_{L^\infty}^2 \tag{5.23}$$

$$\epsilon^3 \left\| \langle \vec{v} \rangle^\vartheta e^{\zeta |\vec{v}|^2} \Gamma[R_N, R_N] \right\|_{L^\infty} \leq C \epsilon^3 \left\| \langle \vec{v} \rangle^\vartheta e^{\zeta |\vec{v}|^2} R_N \right\|_{L^\infty}^2. \tag{5.24}$$

Then if $N \geq 3$, we obtain

$$\left\| \langle \vec{v} \rangle^\vartheta e^{\zeta |\vec{v}|^2} R_N \right\|_{L^\infty} + \left| \langle \vec{v} \rangle^\vartheta e^{\zeta |\vec{v}|^2} R_N \right|_{L^\infty_+} \leq C \epsilon^{N-5} + C \epsilon \left\| \langle \vec{v} \rangle^\vartheta e^{\zeta |\vec{v}|^2} R_N \right\|_{L^\infty}^2, \tag{5.25}$$

which further implies

$$\left\| \langle \vec{v} \rangle^\vartheta e^{\zeta |\vec{v}|^2} R_N \right\|_{L^\infty} + \left| \langle \vec{v} \rangle^\vartheta e^{\zeta |\vec{v}|^2} R_N \right|_{L^\infty_+} \leq \frac{C}{\epsilon^2}. \tag{5.26}$$

for ϵ sufficiently small. This means we have shown

$$\frac{1}{\epsilon^3} \left\| \langle \vec{v} \rangle^\vartheta e^{\zeta |\vec{v}|^2} \left(f^\epsilon - \sum_{k=1}^N \epsilon^k \mathcal{F}_k^\epsilon - \sum_{k=1}^N \epsilon^k \mathcal{F}_k^\epsilon \right) \right\|_{L^\infty} = O\left(\frac{1}{\epsilon^2}\right), \tag{5.27}$$

which naturally leads to the desired result. \square

Hence, combining the estimate of steady Navier-Stokes-type equations and ϵ -Milne problem, we have

$$f^\epsilon = \epsilon^3 R_N + \mathcal{Q}_N + \mathcal{Q}_N, \tag{5.28}$$

exists and is well-posed. The uniqueness and positivity follows from a standard argument as in [8].

6. COUNTEREXAMPLE FOR CLASSICAL APPROACH

In this section, we present the classical approach with the idea in [16, 17] to construct asymptotic expansion, especially the boundary layer expansion, and provide counterexamples to show this method is problematic.

6.1. Interior Expansion. Basically, the expansion for interior solution is identical to our method, so we omit the details and only present the notation. We define the interior expansion

$$\mathcal{F} \sim \sum_{k=1}^{\infty} \epsilon^k \mathcal{F}_k(\vec{x}, \vec{v}), \tag{6.1}$$

with

$$\mathcal{F}_k(\vec{x}, \vec{v}) = A_k(\vec{x}, \vec{v}) + B_k(\vec{x}, \vec{v}) + C_k(\vec{x}, \vec{v}), \tag{6.2}$$

where

$$A_k(\vec{x}, \vec{v}) = \sqrt{\mu} \left(A_{k,0}(\vec{x}) + A_{k,1}(\vec{x}) v_1 + A_{k,2}(\vec{x}) v_2 + A_{k,3}(\vec{x}) \left(\frac{|\vec{v}|^2 - 2}{2} \right) \right), \tag{6.3}$$

$$B_k(\vec{x}, \vec{v}) = \sqrt{\mu} \left(B_{k,0}(\vec{x}) + B_{k,1}(\vec{x})v_1 + B_{k,2}(\vec{x})v_2 + B_{k,3}(\vec{x}) \left(\frac{|\vec{v}|^2 - 2}{2} \right) \right), \quad (6.4)$$

with B_k depending on $A_{s,i}$ in $1 \leq s \leq k-1$ and $i = 0, 1, 2, 3$ as

$$B_{k,0} = 0, \quad (6.5)$$

$$B_{k,1} = \sum_{i=1}^{k-1} A_{i,0} A_{k-i,1}, \quad (6.6)$$

$$B_{k,2} = \sum_{i=1}^{k-1} A_{i,0} A_{k-i,2}, \quad (6.7)$$

$$B_{k,3} = \sum_{i=1}^{k-1} \left(A_{i,0} A_{k-i,3} + A_{i,1} A_{k-i,1} + A_{i,2} A_{k-i,2} \right. \\ \left. + \sum_{j=1}^{k-1-i} A_{i,0} (A_{j,1} A_{k-i-j,1} + A_{j,2} A_{k-i-j,2}) \right), \quad (6.8)$$

and $C_k(\vec{x}, \vec{v})$ satisfies

$$\int_{\mathbb{R}^2} \sqrt{\mu(\vec{v})} C_k(\vec{x}, \vec{v}) \begin{pmatrix} 1 \\ \vec{v} \\ |\vec{v}|^2 \end{pmatrix} d\vec{v} = 0, \quad (6.9)$$

with

$$\mathcal{L}[C_k] = -\vec{v} \cdot \nabla_x \mathcal{F}_{k-1}^\epsilon + \sum_{i=1}^{k-1} \Gamma[\mathcal{F}_i^\epsilon, \mathcal{F}_{k-i}^\epsilon], \quad (6.10)$$

which can be solved explicitly at any fixed \vec{x} . We define

$$A_k = \sqrt{\mu} \left(\rho_k + u_{k,1}v_1 + u_{k,2}v_2 + \theta_k \left(|\vec{v}|^2 - 1 \right) \right), \quad (6.11)$$

Then A_k satisfies the equations as follows:

0th order equations:

$$P_1 - (\rho_1 + \theta_1) = 0, \quad (6.12)$$

$$\nabla_x P_1 = 0, \quad (6.13)$$

1st order equations:

$$P_2 - (\rho_2 + \theta_2 + \rho_1 \theta_1) = 0, \quad (6.14)$$

$$\vec{u} \cdot \nabla_x \vec{u}_1 - \gamma_1 \Delta_x \vec{u}_1 + \nabla_x P_2 = 0, \quad (6.15)$$

$$\nabla_x \cdot \vec{u}_1 = 0, \quad (6.16)$$

$$\vec{u}_1 \cdot \nabla_x \theta_1 - \gamma_2 \Delta_x \theta_1 = 0, \quad (6.17)$$

k^{th} order equations:

$$P_{k+1} - \left(\rho_{k+1} + \theta_{k+1} + \sum_{i=1}^{k+1-i} \rho_i \theta_{k+1-i} \right) = 0, \quad (6.18)$$

$$\sum_{i=1}^k \vec{u}_i \cdot \nabla_x \vec{u}_{k+1-i} - \gamma_1 \Delta_x \vec{u}_k + \nabla_x P_{k+1} = G_{k,1}, \quad (6.19)$$

$$\nabla_x \cdot \vec{u}_k = G_{k,2}, \quad (6.20)$$

$$\sum_{i=1}^k \vec{u}_i \cdot \nabla_x \theta_{k+1-i} - \gamma_2 \Delta_x \theta_k = G_{k,3}, \quad (6.21)$$

where

$$G_{k,j} = G_{k,j}[\vec{x}, \vec{v}; \rho_1, \dots, \rho_{k-1}; \theta_1, \dots, \theta_{k-1}; \vec{u}_1, \dots, \vec{u}_{k-1}], \quad (6.22)$$

is explicit functions depending on lower order terms, and γ_1 and γ_2 are two positive constants.

6.2. Boundary Layer Expansion. By the idea in [16, 17], the boundary layer expansion can be defined by introducing substitutions (3.29), (3.31) and (3.34). Note that we terminate here and do not further use substitution (3.36). Hence, we have transformed the equation (1.12) into

$$\begin{cases} (\vec{v}_r \cdot \vec{n}) \frac{\partial f^\epsilon}{\partial \eta} - \frac{\epsilon}{1 - \epsilon \eta} (\vec{v}_r \cdot \vec{\tau}) \frac{\partial f^\epsilon}{\partial \phi} + \mathcal{L}[f^\epsilon] = \Gamma[f^\epsilon, f^\epsilon], \\ f^\epsilon(0, \phi, \vec{v}_r) = b^\epsilon(0, \phi, \vec{v}_r) \quad \text{for } \vec{v}_r \cdot \vec{n} > 0. \end{cases} \quad (6.23)$$

We define the boundary layer expansion

$$\mathcal{F} \sim \sum_{k=1}^{\infty} \epsilon^k \mathcal{F}_k(\eta, \phi, \vec{v}_r), \quad (6.24)$$

where \mathcal{F}_k can be determined by plugging it into the equation (6.23) and comparing the order of ϵ . Thus in a neighborhood of the boundary, we have

$$(\vec{v}_r \cdot \vec{n}) \frac{\partial \mathcal{F}_1}{\partial \eta} + \mathcal{L}[\mathcal{F}_1] = 0, \quad (6.25)$$

$$(\vec{v}_r \cdot \vec{n}) \frac{\partial \mathcal{F}_2}{\partial \eta} + \mathcal{L}[\mathcal{F}_2] = \frac{1}{1 - \epsilon \eta} (\vec{v}_r \cdot \vec{\tau}) \frac{\partial \mathcal{F}_1}{\partial \phi} + \Gamma[\mathcal{F}_1, \mathcal{F}_1] + 2\Gamma[\mathcal{F}_1, \mathcal{F}_1], \quad (6.26)$$

...

$$(\vec{v}_r \cdot \vec{n}) \frac{\partial \mathcal{F}_k}{\partial \eta} + \mathcal{L}[\mathcal{F}_k] = \frac{1}{1 - \epsilon \eta} (\vec{v}_r \cdot \vec{\tau}) \frac{\partial \mathcal{F}_{k-1}}{\partial \phi} + \sum_{i=1}^{k-1} \Gamma[\mathcal{F}_i, \mathcal{F}_{k-i}] + 2 \sum_{i=1}^{k-1} \Gamma[\mathcal{F}_i, \mathcal{F}_{k-i}]. \quad (6.27)$$

The bridge between the interior solution and boundary layer is the boundary condition

$$f^\epsilon(\vec{x}_0, \vec{v}) = b^\epsilon(\vec{x}_0, \vec{v}). \quad (6.28)$$

Plugging the combined expansion

$$f^\epsilon \sim \sum_{k=1}^{\infty} \epsilon^k (\mathcal{F}_k + \mathcal{F}_k), \quad (6.29)$$

into the boundary condition and comparing the order of ϵ , we obtain

$$\mathcal{F}_1 + \mathcal{F}_1 = b_1, \quad (6.30)$$

$$\mathcal{F}_2 + \mathcal{F}_2 = b_2, \quad (6.31)$$

...

$$\mathcal{F}_k + \mathcal{F}_k = b_k. \quad (6.32)$$

This is the boundary conditions \mathcal{F}_k and \mathcal{F}_k need to satisfy.

6.3. Classical Approach to Construct Asymptotic Expansion. We divide the construction of asymptotic expansion into several steps for each $k \geq 1$:

Step 1: ϵ -Milne Problem.

We solve the ϵ -Milne problem

$$\left\{ \begin{array}{l} (\vec{v}_r \cdot \vec{n}) \frac{\partial g_k}{\partial \eta} + \mathcal{L}[g_k] = S_k(\eta, \phi, \vec{v}_r), \\ g_k(0, \phi, \vec{v}_r) = h_k(\phi, \vec{v}_r) \text{ for } \vec{v}_r \cdot \vec{n} > 0, \\ \int_{\mathbb{R}^2} (\vec{v}_r \cdot \vec{n}) \sqrt{\mu} g_k(0, \phi, \vec{v}_r) d\vec{v}_r = m_f[g_k](\phi), \\ \lim_{\eta \rightarrow \infty} g_k(\eta, \phi, \vec{v}_r) = g_k(\infty, \phi, \vec{v}_r), \end{array} \right. \quad (6.33)$$

for $g_k(\eta, \phi, \vec{v}_r)$ with the in-flow boundary data

$$h_k = b_k - (B_k + C_k) \quad (6.34)$$

and source term

$$S_k = \frac{\Upsilon(\sqrt{\epsilon}\eta)}{1 - \epsilon\eta} (\vec{v}_r \cdot \vec{\tau}) \frac{\partial \mathcal{F}_{k-1}}{\partial \phi} + \sum_{i=1}^{k-1} \Gamma[\mathcal{F}_i, \mathcal{F}_{k-i}] + 2 \sum_{i=1}^{k-1} \Gamma[\mathcal{F}_i, \mathcal{F}_{k-i}], \quad (6.35)$$

where

$$\Upsilon(z) = \begin{cases} 1 & 0 \leq z \leq 1/2, \\ 0 & 3/4 \leq z \leq \infty. \end{cases} \quad (6.36)$$

Here the mass-flux $m_f[g_k](\phi)$ will be determined later. Based on Remark 4.16, there exist

$$\tilde{h}_k = \sqrt{\mu} \left(\tilde{D}_{k,0} + \tilde{D}_{k,1} v_{r,1} + \tilde{D}_{k,2} v_{r,2} + \tilde{D}_{k,3} \left(\frac{|\vec{v}_r|^2 - 2}{2} \right) \right), \quad (6.37)$$

such that the problem

$$\left\{ \begin{array}{l} (\vec{v}_r \cdot \vec{n}) \frac{\partial \mathcal{G}_k}{\partial \eta} + \mathcal{L}[\mathcal{G}_k] = S_k(\eta, \phi, \vec{v}_r), \\ \mathcal{G}_k(0, \phi, \vec{v}_r) = h_k(\phi, \vec{v}_r) - \tilde{h}_k(\phi, \vec{v}_r) \text{ for } \vec{v}_r \cdot \vec{n} > 0, \\ \int_{\mathbb{R}^2} (\vec{v}_r \cdot \vec{n}) \sqrt{\mu} \mathcal{G}_k(0, \phi, \vec{v}_r) d\vec{v}_r = m_f[\mathcal{G}_k](\phi), \\ \lim_{\eta \rightarrow \infty} \mathcal{G}_k(\eta, \phi, \vec{v}_r) = 0, \end{array} \right. \quad (6.38)$$

is well-posed.

Step 2: Definition of Interior Solution and Boundary Layer with Geometric Correction.

Define

$$\mathcal{F}_k = \mathcal{G}_k \cdot \Upsilon_0(\epsilon^{1/2}\eta) \quad (6.39)$$

where \mathcal{G}_k the solution of ϵ -Milne problem (6.38) and

$$\Upsilon_0(z) = \begin{cases} 1 & 0 \leq z \leq 1/4, \\ 0 & 1/2 \leq z \leq \infty. \end{cases} \quad (6.40)$$

Naturally, we have

$$\lim_{\eta \rightarrow 0} \mathcal{F}_k(\eta, \phi, \vec{v}_r) = 0. \quad (6.41)$$

The interior solution

$$\mathcal{F}_k(\vec{x}, \vec{v}) = A_k(\vec{x}, \vec{v}) + B_k(\vec{x}, \vec{v}) + C_k(\vec{x}, \vec{v}), \quad (6.42)$$

where A_k satisfies

$$A_k = \sqrt{\mu} \left(\rho_k + u_{k,1} v_1 + u_{k,2} v_2 + \theta_k \left(\frac{|\vec{v}|^2 - 1}{2} \right) \right), \quad (6.43)$$

and

$$P_{k+1} - \left(\rho_{k+1} + \theta_{k+1} + \sum_{i=1}^{k+1-i} \rho_i \theta_{k+1-i} \right) = 0, \quad (6.44)$$

$$\sum_{i=1}^k \vec{u}_i \cdot \nabla_x \vec{u}_{k+1-i} - \gamma_1 \Delta_x \vec{u}_k + \nabla_x P_{k+1} = H_{k,1}, \quad (6.45)$$

$$\nabla_x \cdot \vec{u}_k = H_{k,2}, \quad (6.46)$$

$$\sum_{i=1}^k \vec{u}_i \cdot \nabla_x \theta_{k+1-i} - \gamma_2 \Delta_x \theta_k = H_{k,3}, \quad (6.47)$$

with boundary condition

$$A_{k,0} = \tilde{D}_{k,0}, \quad (6.48)$$

$$A_{k,1} = -\tilde{D}_{k,0}, \quad (6.49)$$

$$A_{k,2} = -\tilde{D}_{k,0}, \quad (6.50)$$

$$A_{k,3} = \tilde{D}_{k,3}. \quad (6.51)$$

where $\tilde{D}_{k,i}$ comes from the boundary data of Milne problem \tilde{h}_k . This determines $A_{k,0}$, $A_{k,1}$, $A_{k,2}$ and $A_{k,3}$. Now it is easy to verify the boundary data are satisfied as

$$\mathcal{F}_k + \mathcal{F}_k = b_k. \quad (6.52)$$

Step 3: Boussinesq relation and Vanishing Mass-Flux.

Similarly, the free mass-flux $m_f[\mathcal{G}_k](\phi)$ can help to enforce two relations: the Boussinesq relation

$$\rho_k + \theta_k = E_k - \sum_{i=1}^{k-i} \rho_i \theta_{k-i}, \quad (6.53)$$

and vanishing mass-flux relation

$$\int_{\partial\Omega} \int_{\mathbb{R}^2} \mathcal{F}_k(\vec{x}, \vec{v}) d\vec{v} d\gamma = 0. \quad (6.54)$$

Therefore, $m_f[\mathcal{G}_k](\phi)$ is completely determined and so are \mathcal{F}_k and \mathcal{F}_k .

The analysis in [16, 17] anticipates this process can be generalized to arbitrary k . However, In order to show the hydrodynamic limit, we at least need to expand to $k = 2$ at least. Therefore, based on Remark 4.16, we require $S_2 \in L^\infty$ to obtain a well-posed \mathcal{F}_2 , i.e. we need

$$\frac{\partial \mathcal{F}_1}{\partial \phi} \in L^\infty, \quad (6.55)$$

which further requires

$$\frac{\partial \mathcal{F}_1}{\partial \eta} \in L^\infty. \quad (6.56)$$

Theorem 6.1 states that for certain boundary data B^ϵ , this is invalid. Hence, this formulation breaks down.

6.4. Singularity in Derivative of Milne Problem. Now we present the singularity of the normal derivative in the Milne problem. For convenience, we use the notation $\vec{v} = (v_\eta, v_\phi)$.

Theorem 6.1. *For the Milne problem*

$$\left\{ \begin{array}{ll} v_\eta \frac{\partial g}{\partial \eta} + \mathcal{L}[g] &= 0, \\ g(0, \vec{v}) &= h(\vec{v}) \text{ for } v_\eta > 0, \\ \int_{\mathbb{R}^2} v_\eta \sqrt{\mu} g(0, \vec{v}) d\vec{v} &= 0, \\ \lim_{\eta \rightarrow \infty} g(\eta, \vec{v}) &= g_\infty(\vec{v}). \end{array} \right. \quad (6.57)$$

with

$$h = v_\phi e^{-(v_\phi^2 - 1) - M v_\eta^2}, \quad (6.58)$$

where v_η and v_ϕ are defined as in (3.34) and M is sufficiently large such that

$$h(0, 1) = 1, \quad (6.59)$$

$$|h|_{L^2_-} < 1, \quad (6.60)$$

then we have

$$\left\| \frac{\partial g}{\partial \eta} \right\|_{L^\infty} \notin L^\infty([0, \infty) \times \mathbb{R}^2). \quad (6.61)$$

Proof. We divide the proof into several steps: We first assume $\partial_\eta g \in L^\infty([0, \infty) \times \mathbb{R}^2)$ and then show it can lead to a contradiction.

Step 1: Definition of trace.

It is easy to see $\partial_\eta g$ satisfies the Milne problem

$$v_\eta \frac{\partial(\partial_\eta g)}{\partial \eta} + \mathcal{L}[\partial_\eta g] = 0. \quad (6.62)$$

Since $k(\vec{u}, \vec{v}) = k_2(\vec{u}, \vec{v}) - k_1(\vec{u}, \vec{v})$ is in L^1 with respect to \vec{u} uniformly in \vec{v} , then we have $K[\partial_\eta g] \in L^\infty([0, \infty) \times \mathbb{R}^2)$. For fixed $N > 0$, $\nu(\vec{v})$ is bounded in the domain $S = \{|\vec{v}| \leq N\}$. Hence, we have $\nu(\vec{v})\partial_\eta g \in L^\infty([0, \infty) \times S)$, which further implies $\mathcal{L}[\partial_\eta g] \in L^\infty([0, \infty) \times S)$. Therefore, by a standard cut-off argument and Ukai's trace theorem, we deduce $\partial_\eta g(0) \in L^\infty(S)$ is well-defined.

However, we can define the trace of $\partial_\eta g$ in another fashion. For any $v_\eta \neq 0$, since we have $\nu(\vec{v})g \in L^\infty([0, \infty) \times S)$ as well as $K[g] \in L^\infty[0, \infty) \times S$, by the Milne problem (6.57), it is naturally to define for $\eta > 0$

$$\partial_\eta g(\eta, \vec{v}) = \frac{K[g](\eta, \vec{v}) - \nu g(\eta, \vec{v})}{v_\eta}. \quad (6.63)$$

Since $\partial_\eta g \in L^\infty([0, \infty) \times S)$, we know g is continuous with respect to η for a.e. \vec{v} . Taking $\eta \rightarrow 0$ defines the trace for $\partial_\eta g$ at $(0, \vec{v})$

$$\partial_\eta g(0, \vec{v}) = \frac{K[g](0, \vec{v}) - \nu g(0, \vec{v})}{v_\eta}. \quad (6.64)$$

Since the grazing set $\{\vec{v} : v_\eta = 0\}$ is zero-measured on the boundary $\eta = 0$, then we have the trace of $\partial_\eta g$ is a.e. well-defined.

By the uniqueness of trace of $\partial_\eta g$, above two types of traces must coincide with each other a.e.. Then we may combine them both and obtain $\partial_\eta g(0, \vec{v}) \in L^\infty(S)$ is a.e. well-defined and satisfies the formula

$$\partial_\eta g(0, \vec{v}) = \frac{K[g](0, \vec{v}) - \nu g(0, \vec{v})}{v_\eta}. \quad (6.65)$$

Step 2: Limiting Process.

Therefore, we may consider the limiting process

$$\lim_{\vec{v} \rightarrow (0, 1)} \frac{\partial g}{\partial \eta}(0, \vec{v}) = \lim_{\vec{v} \rightarrow (0, 1)} \frac{K[g](0, \vec{v}) - \nu g(0, \vec{v})}{v_\eta}. \quad (6.66)$$

Based on [10, Lemma 3.3.1], we have

$$\|K[g](0)\|_{L^\infty_{0,0}} \leq \|K[g]\|_{L^\infty L^\infty_{0,0}} \leq C\|g\|_{L^\infty L^2_0}. \quad (6.67)$$

By Lemma 4.11, we have

$$\|g\|_{L^\infty L^2_0} \leq C(\delta)\|g\|_{L^2 L^2} + \delta\|g\|_{L^\infty L^\infty_{\vec{\partial}, 0}}, \quad (6.68)$$

for $\delta > 0$ sufficiently small and $\vartheta > 2$. Combining this with Theorem 4.7 and Theorem 4.14, we know

$$\|g\|_{L^2 L^2} \leq C \|h\|_{L^2} \quad (6.69)$$

$$\|g\|_{L^\infty L_{\vartheta,0}^\infty} \leq C < \infty. \quad (6.70)$$

Taking δ sufficiently small, and then taking M sufficiently large, we have

$$\|g\|_{L^\infty L_0^2} < 1. \quad (6.71)$$

On the other hand, we can see

$$\nu(0,1)g(0,0,1) \geq Ch(0,1) \geq C_0 > 0, \quad (6.72)$$

for some positive constant C_0 . Therefore, we have shown

$$\mathcal{L}[g](0, \vec{v}) \geq \frac{C_0}{2} > 0, \quad (6.73)$$

when $\vec{v} \rightarrow (0,1)$ which implies $v_\eta \rightarrow 0$. Hence, we can solve the normal derivative as

$$\frac{\partial g}{\partial \eta} = -\frac{\mathcal{L}[g](0)}{v_\eta} \rightarrow \infty, \quad (6.74)$$

which contradicts our assumption that $\partial_\eta g(0, \vec{v}) \in L^\infty(S)$. \square

6.5. Counterexample to Classical Approach. We present a counterexample to show this classical approach can lead to wrong result.

Theorem 6.2. *For given $B^\epsilon > 0$ satisfying (1.10) with*

$$\frac{b_1}{\sqrt{\mu}} = \left(v_\phi e^{-(v_\phi^2 - 1) - M v_\eta^2} \right) = h(v_\eta, v_\phi), \quad (6.75)$$

where v_η and v_ϕ are defined as in (3.34) and we take M sufficiently large such that

$$h(0,1) = 1, \quad (6.76)$$

$$|h|_{L^2_-} < 1, \quad (6.77)$$

there exists $C > 0$ such that

$$\|f^\epsilon - (\mathcal{F}_1 + \mathcal{F}_1)\|_{L^\infty} \geq C\epsilon, \quad (6.78)$$

where the interior solution \mathcal{F}_1 is defined in (6.44) and boundary layer \mathcal{F}_1 is defined in (6.38).

Proof. Define $\mathcal{W}^\epsilon = \mathcal{F}_1^\epsilon + \mathcal{F}_1^\epsilon$ and $\mathcal{W} = \mathcal{F}_1 + \mathcal{F}_1$. Consider the ϵ -Milne problem

$$\left\{ \begin{array}{l} v_\eta \frac{\partial \mathcal{W}^\epsilon}{\partial \eta} + G(\epsilon; \eta) \left(v_\phi^2 \frac{\partial \mathcal{W}^\epsilon}{\partial v_\eta} - v_\eta v_\phi \frac{\partial \mathcal{W}^\epsilon}{\partial v_\phi} \right) + \mathcal{L}[\mathcal{W}^\epsilon] = 0, \\ \mathcal{W}^\epsilon(0, v_\eta, v_\phi) = h(v_\eta, v_\phi) \text{ for } v_\eta > 0, \\ \int_{\mathbb{R}^2} v_\eta \sqrt{\mu} \mathcal{W}^\epsilon(0, v_\eta, v_\phi) dv_\eta dv_\phi = m_f(\phi), \\ \lim_{\eta \rightarrow \infty} \mathcal{W}^\epsilon(\eta, v_\eta, v_\phi) = \mathcal{W}_\infty^\epsilon(v_\eta, v_\phi), \end{array} \right. \quad (6.79)$$

and Milne problem

$$\left\{ \begin{array}{l} v_\eta \frac{\partial \mathcal{W}}{\partial \eta} + \mathcal{L}[\mathcal{W}] = 0, \\ \mathcal{W}(0, v_\eta, v_\phi) = h(v_\eta, v_\phi) \text{ for } v_\eta > 0, \\ \int_{\mathbb{R}^2} v_\eta \sqrt{\mu} \mathcal{W}(0, v_\eta, v_\phi) dv_\eta dv_\phi = m_f(\phi), \\ \lim_{\eta \rightarrow \infty} \mathcal{W}(\eta, v_\eta, v_\phi) = \mathcal{W}_\infty(v_\eta, v_\phi). \end{array} \right. \quad (6.80)$$

For convenience, we use the same velocity variables. Note that \mathcal{W}^ϵ actually satisfies an ϵ -Milne problem with non-trivial source term. However, based on the proof of Theorem 4.7, this source term will add a $O(\epsilon^{1/2})$ perturbation to \mathcal{W}^ϵ , so we can omit it and concentrate on above simpler form.

We divide the proof into several steps:

Step 1: Continuity of $K[\mathcal{W}^\epsilon]$ and $K[\mathcal{W}]$ at $\eta = 0$.

For any $R_0 > r_0 > 0$ and $\vec{u} = (u_\eta, u_\phi)$, we have

$$\begin{aligned} & |K[\mathcal{W}](0, \vec{v}) - K[\mathcal{W}](\eta, \vec{v})| \\ & \leq \int_{u_\eta \leq r_0} |k(\vec{u}, \vec{v})| |\mathcal{W}(0, \vec{u}) - \mathcal{W}(\eta, \vec{u})| d\vec{u} + \int_{u_\eta \geq R_0} |k(\vec{u}, \vec{v})| |\mathcal{W}(0, \vec{u}) - \mathcal{W}(\eta, \vec{u})| d\vec{u} \\ & \quad + \int_{r_0 \leq u_\eta \leq R_0} |k(\vec{u}, \vec{v})| |\mathcal{W}(0, \vec{u}) - \mathcal{W}(\eta, \vec{u})| d\vec{u} \end{aligned} \quad (6.81)$$

Since we know $\mathcal{W} \in L^\infty([0, \infty) \times \mathbb{R}^2)$, then for any $\delta > 0$ we can take r_0 sufficiently small such that

$$\int_{u_\eta \leq r_0} |k(\vec{u}, \vec{v})| |\mathcal{W}(0, \vec{u}) - \mathcal{W}(\eta, \vec{u})| d\vec{u} \leq C \int_{u_\eta \leq r_0} |k(\vec{u}, \vec{v})| d\vec{u} \leq \frac{\delta}{3}. \quad (6.82)$$

Since we know

$$\left\| \langle \vec{v} \rangle^\vartheta e^{\zeta |\vec{v}|^2} (\mathcal{W} - \mathcal{W}_\infty) \right\|_{L^\infty} \leq C < \infty, \quad (6.83)$$

then there exists a $R_0 > 0$, such that for $u_\eta \geq R_0$,

$$|\mathcal{W}(\eta, \vec{u})| \leq \tilde{\delta}, \quad (6.84)$$

where $\tilde{\delta}$ is sufficiently small. Therefore, we have

$$\int_{u_\eta \geq R_0} |k(\vec{u}, \vec{v})| |\mathcal{W}(0, \vec{u}) - \mathcal{W}(\eta, \vec{u})| d\vec{u} \leq 2\tilde{\delta} \int_{u_\eta \geq R_0} |k(\vec{u}, \vec{v})| d\vec{u} \leq \frac{\delta}{3}. \quad (6.85)$$

For fixed r_0 and R_0 satisfying above requirement, we estimate the integral on $r_0 \leq u_\eta \leq R_0$. By Ukai's trace theorem, we have $\mathcal{W}(0, \vec{v})$ is well-defined and

$$\partial_\eta \mathcal{W}(0, \vec{u}) = \frac{K[\mathcal{W}](0, \vec{v}) - \nu(\vec{v}) \mathcal{W}(0, \vec{v})}{v_\eta}. \quad (6.86)$$

The in $r_0 \leq u_\eta \leq R_0$, $\partial_\eta \mathcal{W}$ is bounded, which implies $\mathcal{W}(\eta, \vec{v})$ is uniformly continuous at $\eta = 0$. Then there exists a η_0 such that for $0 \leq \eta \leq \eta_0$,

$$\int_{r_0 \leq u_\eta \leq R_0} |k(\vec{u}, \vec{v})| |\mathcal{W}(0, \vec{u}) - \mathcal{W}(\eta, \vec{u})| d\vec{u} \leq C\tilde{\delta} \int_{r_0 \leq u_\eta \leq R_0} |k(\vec{u}, \vec{v})| d\vec{u} \leq \frac{\delta}{3}. \quad (6.87)$$

In summary, we have shown for any $\delta > 0$, there exists a $\eta_0 > 0$ such that for any $0 \leq \eta \leq \eta_0$ and fixed \vec{v} ,

$$|K[\mathcal{W}](0, \vec{v}) - K[\mathcal{W}](\eta, \vec{v})| \leq \delta. \quad (6.88)$$

Therefore, $K[\mathcal{W}]$ is continuous at $\eta = 0$. A similar argument can be implemented to \mathcal{W}^ϵ . It is easy to see above estimate is uniform in \vec{v} since L^1 estimate of $k(\vec{u}, \vec{v})$ in \vec{u} is uniform with respect to \vec{v} . Also, it is obvious to see K is continuous with respect to \vec{v} at $\eta = 0$.

Step 2: Milne formulation.

We consider the solution at a specific point $\eta = n\epsilon$, $v_\eta = \epsilon$ and $v_\phi = \sqrt{1 - \epsilon^2}$ for some fixed $n > 0$. The solution along the characteristics can be rewritten as follows:

$$\mathcal{W}(n\epsilon, \epsilon, \sqrt{1 - \epsilon^2}) = h(\epsilon, \sqrt{1 - \epsilon^2}) e^{-\frac{\nu(1)}{\epsilon} n\epsilon} + \int_0^{n\epsilon} e^{-\frac{\nu(1)}{\epsilon} (n\epsilon - \kappa)} \frac{1}{\epsilon} K[\mathcal{W}](\kappa, \epsilon, \sqrt{1 - \epsilon^2}) d\kappa, \quad (6.89)$$

$$(6.90)$$

$$\mathcal{W}^\epsilon(n\epsilon, \epsilon, \sqrt{1 - \epsilon^2}) = h(\epsilon_0, \sqrt{1 - \epsilon_0^2}) e^{-\int_0^{n\epsilon} \frac{\nu(1)}{v_\eta(\zeta)} d\zeta} + \int_0^{n\epsilon} e^{-\int_\kappa^{n\epsilon} \frac{\nu(1)}{v_\eta(\zeta)} d\zeta} \frac{1}{v_\eta(\kappa)} K[\mathcal{W}^\epsilon](\kappa, v_\eta(\kappa), v_\phi(\kappa)) d\kappa,$$

where $\nu(1)$ denote the value of $\nu(\vec{v})$ at $|\vec{v}| = 1$ and we have the conserved energy along the characteristics

$$E(\eta, v_\eta, v_\phi) = v_\phi e^{-W(\eta)}, \quad (6.91)$$

in which $(0, \epsilon_0, \sqrt{1 - \epsilon_0^2})$ and $(\zeta, v_\eta(\zeta), \sqrt{1 - v_\eta^2(\zeta)})$ are in the same characteristics of $(n\epsilon, \epsilon, \sqrt{1 - \epsilon^2})$.

Step 3: Estimates of (6.89).

We turn to the Milne problem for \mathcal{W} . We have the natural estimate

$$\int_0^{n\epsilon} e^{-\frac{\nu(1)}{\epsilon}(n\epsilon-\kappa)} \frac{1}{\epsilon} d\kappa = e^{-n\nu(1)} \int_0^{n\epsilon} e^{\frac{\nu(1)\kappa}{\epsilon}} \frac{1}{\epsilon} d\kappa = e^{-n\nu(1)} \int_0^n e^{\nu(1)\zeta} d\zeta = \frac{1}{\nu(1)} \left(1 - e^{-n\nu(1)}\right). \quad (6.92)$$

Then for $0 < \epsilon \leq \eta_0$, we have $|K[\mathcal{W}](0, 0, 1) - K[\mathcal{W}](\kappa, \epsilon, \sqrt{1-\epsilon^2})| \leq \delta + \epsilon$, which implies

$$\begin{aligned} \int_0^{n\epsilon} e^{-\frac{\nu(1)}{\epsilon}(n\epsilon-\kappa)} \frac{1}{\epsilon} K[\mathcal{W}](\kappa, \epsilon, \sqrt{1-\epsilon^2}) d\kappa &= \int_0^{n\epsilon} e^{-\frac{\nu(1)}{\epsilon}(n\epsilon-\kappa)} \frac{1}{\epsilon} K[\mathcal{W}](0, 0, 1) d\kappa + O(\delta) + O(\epsilon) \\ &= \frac{1}{\nu(1)} (1 - e^{-n\nu(1)}) K[\mathcal{W}](0, 0, 1) + O(\delta) + O(\epsilon). \end{aligned} \quad (6.93)$$

For the boundary data term, it is easy to see

$$h(\epsilon, \sqrt{1-\epsilon^2}) e^{-\frac{\nu(1)}{\epsilon}n\epsilon} = e^{-n\nu(1)} h(\epsilon, \sqrt{1-\epsilon^2}). \quad (6.94)$$

In summary, we have

$$\mathcal{W}(n\epsilon, \epsilon, \sqrt{1-\epsilon^2}) = \frac{1}{\nu(1)} (1 - e^{-n\nu(1)}) K[\mathcal{W}](0, 0, 1) + e^{-n\nu(1)} h(0, 1) + O(\delta) + O(\epsilon). \quad (6.95)$$

Step 4: Estimates of (6.90).

We consider the ϵ -Milne problem for \mathcal{W}^ϵ . For $\epsilon \ll 1$ sufficiently small, $\psi(\epsilon) = 1$. Then we may estimate

$$v_\phi(\zeta) e^{-W(\zeta)} = \sqrt{1-\epsilon^2} e^{-W(n\epsilon)}, \quad (6.96)$$

which implies

$$v_\phi(\zeta) = \frac{1-n\epsilon^2}{1-\epsilon\zeta} \sqrt{1-\epsilon^2}. \quad (6.97)$$

and hence

$$v_\eta(\zeta) = \sqrt{1-v_\phi^2(\zeta)} = \sqrt{\frac{\epsilon(n\epsilon-\zeta)(2-\epsilon\zeta-n\epsilon^2)}{(1-\epsilon\zeta)^2} (1-\epsilon^2) + \epsilon^2}. \quad (6.98)$$

For $\zeta \in [0, \epsilon]$ and $n\epsilon$ sufficiently small, by Taylor's expansion, we have

$$1 - \epsilon\zeta = 1 + o(\epsilon), \quad (6.99)$$

$$2 - \epsilon\zeta - n\epsilon^2 = 2 + o(\epsilon). \quad (6.100)$$

Hence, we have

$$v_\eta(\zeta) = \sqrt{\epsilon(\epsilon + 2n\epsilon - 2\zeta)} + o(\epsilon^2). \quad (6.101)$$

Since $\sqrt{\epsilon(\epsilon + 2n\epsilon - 2\zeta)} = O(\epsilon)$, we can further estimate

$$\frac{1}{v_\eta(\zeta)} = \frac{1}{\sqrt{\epsilon(\epsilon + 2n\epsilon - 2\zeta)}} + o(1) \quad (6.102)$$

$$- \int_\kappa^{n\epsilon} \frac{\nu(1)}{v_\eta(\zeta)} d\zeta = \nu(1) \sqrt{\frac{\epsilon + 2n\epsilon - 2\zeta}{\epsilon}} \Big|_\kappa^{n\epsilon} + o(\epsilon) = \nu(1) \left(1 - \sqrt{\frac{\epsilon + 2n\epsilon - 2\kappa}{\epsilon}}\right) + o(\epsilon). \quad (6.103)$$

Then we can easily derive the integral estimate

$$\begin{aligned}
 \int_0^{n\epsilon} e^{-\int_\kappa^{n\epsilon} \frac{\nu(1)}{v_\eta(\zeta)} d\zeta} \frac{1}{v_\eta(\kappa)} d\kappa &= e^{\nu(1)} \int_0^{n\epsilon} e^{-\nu(1)\sqrt{\frac{\epsilon+2n\epsilon-2\kappa}{\epsilon}}} \frac{1}{\sqrt{\epsilon(\epsilon+2n\epsilon-2\kappa)}} d\kappa + o(\epsilon) \\
 &= \frac{1}{2} e^{\nu(1)} \int_\epsilon^{(1+2n)\epsilon} e^{-\nu(1)\sqrt{\frac{\sigma}{\epsilon}}} \frac{1}{\sqrt{\epsilon\sigma}} d\sigma + o(\epsilon) \\
 &= \frac{1}{2} e^{\nu(1)} \int_1^{1+2n} e^{-\nu(1)\sqrt{\rho}} \frac{1}{\sqrt{\rho}} d\rho + o(\epsilon) \\
 &= e^{\nu(1)} \int_1^{\sqrt{1+2n}} e^{-\nu(1)t} dt + o(\epsilon) \\
 &= \frac{1}{\nu(1)} (1 - e^{\nu(1)(1-\sqrt{1+2n})}) + o(\epsilon).
 \end{aligned} \tag{6.104}$$

Then for $0 < \epsilon \leq \eta_0$, we have $|K[\mathcal{W}^\epsilon](0, 0, 1) - K[\mathcal{W}^\epsilon](\kappa, v_\eta(\kappa), v_\phi(\epsilon))| \leq \delta$, which implies

$$\begin{aligned}
 &\int_0^{n\epsilon} e^{-\int_\kappa^{n\epsilon} \frac{\nu(1)}{v_\eta(\zeta)} d\zeta} \frac{1}{v_\eta(\kappa)} K[\mathcal{W}^\epsilon](\kappa, v_\eta(\kappa), v_\phi(\epsilon)) d\kappa \\
 &= \int_0^{n\epsilon} e^{-\int_\kappa^{n\epsilon} \frac{\nu(1)}{v_\eta(\zeta)} d\zeta} \frac{1}{v_\eta(\kappa)} K[\mathcal{W}^\epsilon](0, 0, 1) d\kappa + O(\delta) + O(\epsilon) \\
 &= \frac{1}{\nu(1)} (1 - e^{\nu(1)(1-\sqrt{1+2n})}) K[\mathcal{W}^\epsilon](0, 0, 1) + O(\epsilon) + O(\delta).
 \end{aligned} \tag{6.105}$$

For the boundary data term, since $h(v_\eta, v_\phi)$ is C^1 , a similar argument shows

$$h(\epsilon_0, \sqrt{1 - \epsilon_0^2}) e^{-\int_0^{n\epsilon} \frac{\nu(1)}{v_\eta(\zeta)} d\zeta} = e^{\nu(1)(1-\sqrt{1+2n})} h(\sqrt{1+2n}\epsilon, \sqrt{1 - (1+2n)\epsilon^2}) + O(\epsilon). \tag{6.106}$$

Therefore, we have

$$\mathcal{W}^\epsilon(n\epsilon, \epsilon) = \frac{1}{\nu(1)} (1 - e^{\nu(1)(1-\sqrt{1+2n})}) K[\mathcal{W}^\epsilon](0, 0, 1) + e^{\nu(1)(1-\sqrt{1+2n})} h(0, 1) + O(\epsilon) + O(\delta). \tag{6.107}$$

Step 5: Estimate of Difference.

Collecting all above, we can estimate the value at point $\eta = n\epsilon$, $v_\eta = \epsilon$ and $v_\phi = \sqrt{1 - \epsilon^2}$ as

$$\begin{aligned}
 \mathcal{W}^\epsilon(n\epsilon, \epsilon, \sqrt{1 - \epsilon^2}) &= \frac{1}{\nu(1)} (1 - e^{\nu(1)(1-\sqrt{1+2n})}) K[\mathcal{W}^\epsilon](0, 0, 1) + e^{\nu(1)(1-\sqrt{1+2n})} h(0, 1) + O(\epsilon) + O(\delta), \\
 &\tag{6.108}
 \end{aligned}$$

$$\mathcal{W}(n\epsilon, \epsilon, \sqrt{1 - \epsilon^2}) = \frac{1}{\nu(1)} (1 - e^{-\nu(1)n}) K[\mathcal{W}](0, 0, 1) + e^{-\nu(1)n} h(0, 1) + O(\epsilon) + O(\delta). \tag{6.109}$$

By our assumptions on h and a similar argument as in the proof of Theorem 6.1, we know $|K[\mathcal{W}^\epsilon](0, 0, 1)| \ll 1$ and $|K[\mathcal{W}](0, 0, 1)| \ll 1$. However, $h(0, 1) = 1$. Since n is arbitrary and $e^{\nu(1)(1-\sqrt{1+2n})} \neq e^{-\nu(1)n}$, we always have

$$\left| \mathcal{W}^\epsilon(\epsilon, n\epsilon, \sqrt{1 - n^2\epsilon^2}) - \mathcal{W}(\epsilon, n\epsilon, \sqrt{1 - n^2\epsilon^2}) \right| \geq C > 0, \tag{6.110}$$

which further implies

$$\| \mathcal{W}^\epsilon - \mathcal{W} \|_{L^\infty L_{0,0}^\infty} \geq C > 0. \tag{6.111}$$

□

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